Construction and geometric validity (positive jacobian) of serendipity Lagrange finite elements, theory and practical guidance

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Abstract

Finite elements of degree two or more are needed to solve various P.D.E. problems. This paper discusses a method to validate such meshes for the case of the serendipity Lagrange elements of various degree. The first section of this paper comes back to Bézier curve and Bézier patches of arbitrary degree. The way in which a Bézier patch and a complete finite element are related is recalled. The construction of serendipity or reduced Lagrange finite elements of various degree is discussed, including simplices (triangle and tetrahedron), quads, prisms (pentahedron), pyramids and hexes. The validity condition, the positivity of the jacobian, exhibited for the classical (complete) elements is used to validate their serendipity counterparts after having *invented* a complete element equivalent to the reduced element under analyse.

1 Introduction

High order (p-version) finite elements are employed to accurately solve a number of *P.D.E.* with a good rate of convergence, see [2], [3], [9], [10], [27]. The order impacts two different aspects, one concerning the geometry, the other the finite element approximation. These two aspects may be combined or not. For instance, a high order element in the case of a straight-sided geometry does not lead to any difficulty at the time the geometry is considered, while even a not too high order element where the geometry is a curved geometry may lead to some tedious questions, see the pioneering references, [29], [28], [30] and [13] and some more recent references, including [7] for tetrahedral elements, [21] for quadrilaterals and [24] for triangles. In this paper, we are only concerned with the geometric validity of high order serendipity meshes of planar or volume domains with curved boundaries but we are not directly interested in the finite element aspect, e.g. solution methods and mesh quality. As regards the validity of a given mesh, a common idea is that it is sufficient to locate the nodes on the curved edges without giving any explicit attention to the positivity of the resulting jacobian. Another idea and one that is advocated in a number of papers is to evaluate the jacobian on a sample of points (for example Gauss points)

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but this is only a necessary condition. Actually, this works well in most cases but only if the boundary is not too bended.

The idea is to return to the case of the classical Lagrange elements where we have sufficient conditions for the geometric validity. To this end, given a serendipity element, it is required to *invent* a complete element which is equivalent to the reduced element.

Before going to this question, we first discuss about various methods suitable for the effective construction of the serendipity or reduced elements, since there is a shortage of literature about this point (apart for the 8-node quad and the 9-node triangle).

Nomenclature

- $\hat{K}$ the reference element, K the current element, F_K the mapping from $\hat{K}$ to K, $p_i, p_{ij}, ...$, a shape function, d the degree of the finite element, $\mathcal{J}$ the jacobian of K, q the degree of this jacobian,
- $\hat{A}_i$, $A_i, (A_{ij}, A_{ijk}, A_{ijkl})$, a node of $\hat{K}$ and its image by F_K,
- u, v, w, t or $\hat{x}, \hat{y}, \hat{z}$, the parameters living in the parametric space, e.g. $\hat{K}$,
- Γ and γ, Σ and σ, Θ and θ, resp. a curve and its expression, a bidimensional patch and its expression, a tridimensional patch and its expression,
- P_{ij} (P_{ijk}, P_{ijkl}) a control point, N_{ij} (N_{ijk}, N_{ijkl}) a (scalar) control value,
- $B_i^d(u)$ the Bernstein polynomial of degree d for a system of natural coordinates, $B_{ij}^d(u,v), B_{ijk}^d(u,v,w), ...$, the Bernstein polynomial of degree d for a system of *barycentric* coordinates, C_i^d, the binomial coefficient,
- $[.]$ and $|.|$, a matrix and a determinant.

For a further usage, the first three sections recall some basics about Bézier patches (Section 2) and usual (complete) Lagrange finite elements (Sections 3 and 4).

2 Bernstein polynomials, Bézier curves and Bézier patches

Following [5] and [14], a Bézier curve (the edges of the elements) of degree d is defined by means of $d+1$ control points and the Bernstein basis. More precisely, let $P_i \in \mathbb{R}^2$ or R^3 be those points, the curve Γ reads

$$\Gamma = \left\{ \gamma(u) = \sum_{i=0,d} B_i^d(u) P_i \text{ with } u \in [0,1] \right\}, \tag{1}$$

and, using a system of *barycentric* coordinates, the same reads

$$\Gamma = \left\{ \gamma(u,v) = \sum_{i+j=d} B_{ij}^d(u,v) P_{ij}, \text{ with } u+v=1 \right\}. \tag{2}$$

In the above equations, the Bernstein polynomials respectively read as

$$B_i^d(u) = C_i^d u^i (1-u)^{d-i} = \frac{d!}{i!(d-i)!} u^i (1-u)^{d-i}, \tag{3}$$

and

$$B_{ij}^d(u,v) = C_{ij}^d u^i v^j = \frac{d!}{i!j!} u^i v^j \text{ with } i+j=d. \tag{4}$$

and so on.

This abstract reading of $\gamma(u,v)$ or $\gamma(u)$ extends to tensor-product patches, for instance in two dimensions, and for the degree $d \times d$, we have

$$\sigma(u,v) \;=\sum_{i=0,d}\sum_{j=0,d} B_i^d(u)B_j^d(v)\,P_{ij},, \tag{5}$$

and the patch reads as

$$\Sigma \;=\; \{\sigma(u,v)\,\text{with}\,(u,v)\in[0,1]\times[0,1]\}\;, \tag{6}$$

which is an element by itself or a face element in three dimensions. This extends to tridimensional tensor-product patches (e.g. by defining $\theta(u,v,w), \Theta, ...$ accordingly). Such definitions will be used to define quadrilaterals, hexes and quadrilateral faces (in the case of a prism or a pyramid).

As for simplices or triangular faces, it is much more convenient to use the *barycentric* form of the Bézier setting, e.g

$$\Sigma \;=\; \left\{\sigma(u,v,w)\,\text{with}\,u+v+w=1\,,\text{where}\,\sigma(u,v,w)\;=\sum_{i+j+k=d} B_{ijk}^d(u.v,w)\,P_{ijk}\right\}\;, \tag{7}$$

for a triangle or a triangular face and

$$\Theta \;=\; \left\{\theta(u,v,w,t)\,\text{with}\,u+v+w+t=1\,\text{and}\,\theta(u,v,w,t)\;=\sum_{i+j+k+l=d} B_{ijkl}^d(u,v,w,t)\,P_{ijkl}\right\}\;, \tag{8}$$

for a tet patch.

3 Bézier form and finite element form of a complete element

This section briefly recalls the basics of what a finite element is and shows that complete[1] Lagrange elements can be written in terms of a Bézier form. To do this, we follow [9] and more precisely [10], using the same notations.

Let K be a geometric element (triangle, quad, tet, ...), a Lagrange finite element associated with K is defined by the triple $[K, P, Nodes]$ where K is the element, P is a space of polynomials and $Nodes$ is a set of nodes. Actually, K is constructed as the image of a reference element $\hat{K}$, equipped with a set of reference nodes, by means of a mapping F_K, e.g., $K = F_K(\hat{K})$ and, in turn, F_K is defined by means of the polynomials in space P and we have $F_K(\hat{A}) = \sum_{i=0,n-1} p_i(\hat{A})A_i$ where p_i is a polynomial, n is the number of such polynomials (e.g. the dimension of space P), A_i is the node i of K and $\hat{A}$ is the value of the parameters (e.g., for instance, (u,v) or $(\hat{x},\hat{y})$) where F_K is evaluated. Therefore, if we consider K as a patch (such as Σ in the previous section) we have (with evident notations)

$$K = \left\{M(u,v)\,\text{with}\,(u,v)\in\hat{K}\,\text{where}\,M(u,v) = \sum_{i=0,n-1} p_i(u,v)A_i\right\}\;, \tag{9}$$

[1]incomplete element can also be written in this way but it is more subtle.

in other words, the finite element is defined by means of *shape functions* and *nodes*.

Let us consider now a Bézier form like (here is the case of a quadrilateral patch)

$$\sigma(u,v) = \sum_{i=0,d} \sum_{j=0,d} B_i^d(u) B_j^d(v) P_{ij} \,,$$

where d is the degree of space P and P_{ij} is a set of control points, i.e., a patch defined by means of *Bernstein polynomials* and *control points*.

As a matter of fact, for a complete element, space P is complete so that it can be expressed both in terms of the above $p_i(u,v)$ and the Bernstein polynomials. In other words, we have (with appropriate notations)

$$\sum_{i=0,n-1} p_i(u,v) A_i = \sum_{i=0,d} \sum_{j=0,d} B_i^d(u) B_j^d(v) P_{ij} \,.$$

As a consequence the A_is can be written in terms of the P_{ij}s and *vice versa* and the p_is are linear combinations of the B_i^d and *vice versa*.

To simply illustrate this point, we return to the simple case of a Bézier curve of degree 2. It reads

$$\sum_{i=0,2} B_i^2(u) P_i = (1-u)^2 P_0 + 2u(1-u) P_1 + u^2 P_2 \,,$$

let us define $A_0 = P_0, A_1 = \frac{P_0+P_2+2P_1}{4}$ and $A_2 = P_2$, then $P_0 = A_0, P_1 = \frac{-A_0-A_2+4A_1}{2}$ and $P_2 = A_2$ and, replacing the P_is in the above relation, we have

$$(1-u)^2 A_0 + 2u(1-u)\frac{-A_0 - A_2 + 4A_1}{2} + u^2 A_2 \;=\; (1-u)(1-2u)A_0 + 4u(1-u)A_1 + u(2u-1)A_2 \,,$$

which is the classical Lagrange form of the curve. This mechanism applies whatever the degree of the curve and also applies for the patches themselves. The main interest is then to replace the finite elements by their equivalent Bézier setting, making simpler and systematic the calculation and the analysis of their jacobian polynomials.

4 Computing and evaluating the jacobian of a complete element

First of all, we introduce the control points associated with a given element K (formulae allows for this in the case K is defined by its nodes). Then, we write the finite element in its Bézier setting and we express its jacobian. This polynomial being a product of Bernstein polynomials (derivatives are multiplied one each other) is, itself, a Bézier form. Therefore we have immediately a sufficient condition of positiveness: the coefficients of the polynomial must be strictly positive in the case of an interpolant coefficient and non negative if not.

4.1 Complete Lagrange triangle or quad of degree d

The shape functions for Lagrange triangles together with for Lagrange quads can be written using a generic formulation. Let

$$\phi_i(u) = \frac{(-1)^i}{i!(d-i)!} \Pi_{l=0, l\neq i}^{l=d} (l - du) \,,$$

then, the shape function with index $_{ij}$ for a quad of degree $d \times d$ simply reads

$$p_{ij}(u, v) = \phi_i(u)\phi_j(v) \,. \tag{10}$$

The generic expression for a triangle of degree d involves using the system of *barycentric* coordinates. Let

$$\phi_i(u) = \frac{1}{i!}\Pi_{l=0}^{i-1}(du - l) \,,$$

then, the shape function with index $_{ijk}$ for a triangle of degree d simply reads

$$p_{ijk}(u, v, w) = \phi_i(u)\phi_j(v)\phi_k(w) \,. \tag{11}$$

Triangle of degree d. The finite element (as being complete) is written in a Bézier form as

$$\Sigma = \sigma(u, v, w) = \sum_{i+j+k=d} B^d_{ijk}(u, v, w)P_{ijk} \,, \tag{12}$$

where the P_{ijk}s are the control points. The jacobian polynomial reads

$$\mathcal{J}(u, v, w) \, = \sum_{I+J+K=2(d-1)} B^q_{IJK}(u, v, w)N_{IJK} \,,$$

where $q = 2(d-1)$ and the coefficients N_{IJK} are

$$N_{IJK} \, = \, d^2 \sum_{i_1+i_2=I, j_1+j_2=J, k_1+k_2=K} \frac{C^{d-1}_{i_1 j_1 k_1} C^{d-1}_{i_2 j_2 k_2}}{C^q_{IJK}} \left|\Delta^{100}_{i_1+1, j_1 k_1} \quad \Delta^{010}_{i_2+1, j_2 k_2}\right| \,, \tag{13}$$

with $i_1 + j_1 + k_1 = i_2 + j_2 + k_2 = d - 1$, C^q_{ijk} a coefficient (as previously defined) and with the following Δ:

$$\Delta^{100}_{ijk} = \overrightarrow{P_{ijk}P_{i-1,j+1,k}} \quad \text{and} \quad \Delta^{010}_{ijk} = \overrightarrow{P_{ijk}P_{i-1,j,k+1}} \,.$$

The jacobian is a polynomial of degree $q = 2(d-1)$, the number of control coefficients is $\frac{(q+1)\times(q+2)}{2}$ which rapidly increases with q or d as it is for the number of terms (e.g., $(\frac{d\times(d+1)}{2})^2$) to be computed (some coefficients reduce to one term while others are the summation of a number of such terms). The geometry of the observed element is valid if the "corner" coefficients are strictly positive while the others are non negative.

Quad of degree d. The finite element (as being complete) is written in a Bézier form as

$$\sigma(u, v) \, = \sum_{i=0,d} \sum_{j=0,d} B^d_i(u)B^d_j(v) \, P_{ij} \,, \tag{14}$$

where the P_{ij}s are the control points. The jacobian polynomial has the form

$$\mathcal{J}(u, v) \, = \sum_{I=0,q} \sum_{J=0,q} B^q_I(u)B^q_J(v)N_{IJ} \,,$$

where $q = 2d - 1$ and the coefficient N_{IJ}s are

$$N_{IJ} \, = \, d^2 \sum_{i_1+i_2=I} \sum_{j_1+j_2=J} \frac{C^{d-1}_{i_1} C^d_{i_2}}{C^q_I} \frac{C^d_{j_1} C^{d-1}_{j_2}}{C^q_J} |\Delta^{1,0}_{i_1,j_1} \quad \Delta^{0,1}_{i_2,j_2}| \,. \tag{15}$$

$$\text{with}\,\Delta^{1,0}_{i,j} = \overrightarrow{P_{ij}P_{i+1,j}}\,\text{and}\,\Delta^{0,1}_{i,j} = \overrightarrow{P_{ij}P_{i,j+1}}\,,$$

and with $i_1 = 0, d-1, i_2 = 0, d, j_1 = 0, d, j_2 = 0, d-1$ and C^q_{ij} the binomial coefficients. The jacobian is a polynomial of degree $q \times q = (2d-1) \times (2d-1)$, the number of control coefficients is $(q+1)^2$ which rapidly increases with q or d and the number of terms (e.g. determinants) to be computed (some coefficients reduce to one term while others are the summation of a number of such8 terms) is $d^2(d+1)^2$. The geometry of the observed element is valid if the "corner" coefficients are strictly positive while the others are non negative.

5 Bidimensional incomplete Lagrange elements

Incomplete or reduced elements have a reduced number of nodes (typically the edge nodes are those of the complete elements while the number of internal nodes is zero or smaller than that in the complete element). Low degree elements are well documented in the literature, at least for quad geometries (8-node quad of degree 2) and for the 9-node triangle of degree 3. The polynomial space is also of a smaller dimension as compared with the complete space.

There are different methods to define reduced elements among which we have the serendipity elements where space P is rich enough to achieve a good level of precision. One method specifies space P and, given an adequate number of nodes, constructs the shape functions by solving an adequate system satisfying the desired properties (basicaly using the Kronecker delta). Another method makes use of Taylor expansions in order to eliminate the internal nodes. Whatever the method, the shape functions have a generic expression (such as (10) and (11) for the complete Lagrange elements). Let us consider the case of a tensor-product complete element and let p^c_{ij} be its shape functions, then, for the reduced element, we have

$$p_{ij}(u,v) = p^c_{ij}(u,v) + \sum_{kl} \alpha^{kl}_{ij} p^c_{kl}(u,v)\,, \tag{16}$$

where indices ij correspond to the edge[2] nodes and indices kl are those related to the internal[3] nodes of the complete element and α^{kl}_{ij} is a coefficient (of repartition, how p^c_{kl} contributes to p_{ij}). For reduced simplices, we have a similar generic expression

$$p_{ijk}(u,v,w) = p^c_{ijk}(u,v,w) + \sum_{lmn} \alpha^{lmn}_{ijk} p^c_{lmn}(u,v,w)\,. \tag{17}$$

In the rest of the paper, we give a detailed discussion about various methods of construction and, this being done and given a reduced element in a mesh, we discuss the conditions that give guarantees about its geometric validity. As will be seen, the main idea is, given such an element in a mesh, to return to a complete element equivalent to this reduced one and then to apply what we did previously for complete elements. It turns out that this requires to properly *invent* the "missing" nodes and, more precisely, the "missing" control points.

5.1 Order d serendipity (or reduced) triangles

To have at least one internal node, we need to have $d = 3$, so we meet the complete triangle of degree 3, the well-known 10-node triangle, where we have only one internal node.

[2] actually, for some reduced elements, one or several internal nodes of the complete element are retained as nodes for the reduced element.

[3] cf. infra.

The 9-node triangle of degree 3. The numbering of the nodes of the 10-node (9-node) triangles is as follow:

```
           003                          003
        102    012        ==>        102    012
     201    111    021            201           021
  300    210    120    030     300    210    120    030
```

A Taylor expansion, based on the fact that the reduced polynomial space must contain space $P^{d-1} = P^2$ (in terms of the variables x and y, the span of P^2 is made up of $1, x, y, xy, x^2$ and y^2), is used to express the edge values of a generic function (let q be this function) in terms of the internal value of this function. This leads, cf. [4], to a relation[4] like

$$12q(\hat{A}_{111}) + 2\sum_{ijk\in\mathcal{V}} q(\hat{A}_{ijk}) - 3\sum_{ijk\in\mathcal{E}} q(\hat{A}_{ijk}) = 0\,, \tag{18}$$

where $\hat{A}_{111}$ is the point, in the reference element ($\hat{K}$), of coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $\mathcal{V}$ stands for the set of vertices in $\hat{K}$ and $\mathcal{E}$ stands for the set of edge nodes in $\hat{K}$.

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Proof: We consider the template depicted in the above diagram (left side) and we use a Taylor expansion to write the value of a function q in $\hat{A}_{300}, ..., \hat{A}_{201}$, the 9 points in the boundary of the template in terms of its value and its derivatives $D^1.(.), D^2.(.,.)$ evaluated at $\hat{A}_{111}$. Since the expansion terminates at order 2, for a vertex, here A_{300}, we have

$$q(\hat{A}_{300}) = q(\hat{A}_{111}) + D^1.(\overrightarrow{U_{300}}) + D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{300}})\,,$$

where $\overrightarrow{U_{300}} = \overrightarrow{\hat{A}_{111}\hat{A}_{300}}$, the derivatives are evaluated at $\hat{A}_{111}$ and the symbol $D^2.$ includes (for short) the coefficient, here $\frac{1}{2}$. The same holds for the other vertices in $\mathcal{V}$, so we have

$$\sum_{ijk\in\mathcal{V}} q(\hat{A}_{ijk}) = 3q(\hat{A}_{111}) + \sum_{ij\in\mathcal{V}} D^1.(\overrightarrow{U_{ijk}}) + \sum_{ijk\in\mathcal{V}} D^2.(\overrightarrow{U_{ijk}}, \overrightarrow{U_{ijk}})\,. \tag{19}$$

A similar relation holds for the six nodes in $\mathcal{E}$ and we have

$$\sum_{ijk\in\mathcal{E}} q(\hat{A}_{ijk}) = 6q(\hat{A}_{111}) + \sum_{ijk\in\mathcal{E}} D^1.(\overrightarrow{V_{ijk}}) + \sum_{ijk\in\mathcal{E}} D^2.(\overrightarrow{V_{ijk}}, \overrightarrow{V_{ijk}})\,. \tag{20}$$

where $\overrightarrow{V_{ijk}} = \overrightarrow{\hat{A}_{111}\hat{A}_{ijk}}$ with $ijk \in \mathcal{E}$.

The derivative D^1 is a linear operator, so $\sum_{ijk\in\mathcal{V}} D^1.(\overrightarrow{U_{ijk}}) = D^1.(\sum_{ijk\in\mathcal{V}} \overrightarrow{U_{ijk}}) = 0$, similarly we have$\sum_{ijk\in\mathcal{E}} D^1.(\overrightarrow{V_{ijk}}) = 0$. Now, $\overrightarrow{V_{ijk}}$ can be written in terms of the $\overrightarrow{U_{ijk}}$. Indeed, let us consider $\hat{A}_{210}$, we have $\hat{A}_{210} = \frac{2\hat{A}_{300}+\hat{A}_{030}}{3}$ so $\overrightarrow{V_{210}} = \frac{2\overrightarrow{U_{300}}+\overrightarrow{U_{030}}}{3}$ holds. Since, again for index$_{210}$, we have

$$D^2.(\overrightarrow{V_{210}}, \overrightarrow{V_{210}}) = D^2.(\frac{2\overrightarrow{U_{300}} + \overrightarrow{U_{030}}}{3}, \frac{2\overrightarrow{U_{300}} + \overrightarrow{U_{030}}}{3})$$

[4]the so-called serendipity relation.

$$= \frac{4}{9} D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{300}}) + \frac{4}{9} D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{030}}) + \frac{1}{9} D^2.(\overrightarrow{U_{030}}, \overrightarrow{U_{030}}) \,,$$

and, by symmetry

$$D^2.(\overrightarrow{V_{120}}, \overrightarrow{V_{120}}) = \frac{1}{9} D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{300}}) + \frac{4}{9} D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{030}}) + \frac{4}{9} D^2.(\overrightarrow{U_{030}}, \overrightarrow{U_{030}}) \,,$$

and similar expressions for the other edge nodes, we have

$$\sum_{ijk \in \mathcal{E}} D^2.(\overrightarrow{V_{ijk}}, \overrightarrow{V_{ijk}}) = \frac{10}{9} D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{300}}) + \frac{10}{9} D^2.(\overrightarrow{U_{030}}, \overrightarrow{U_{030}}) + \frac{10}{9} D^2.(\overrightarrow{U_{003}}, \overrightarrow{U_{003}})$$

$$+ \frac{8}{9} D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{030}}) + \frac{8}{9} D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{003}}) + \frac{8}{9} D^2.(\overrightarrow{U_{030}}, \overrightarrow{U_{003}}) \,, \tag{21}$$

in other words, the sum for ijk in $\mathcal{E}$ is replaced by sums for ijk in $\mathcal{V}$. Now, because $\sum_{ijk \in \mathcal{V}} \overrightarrow{U_{ijk}} = 0$, we have

$$D^2.(\sum_{ijk \in \mathcal{V}} \overrightarrow{U_{ijk}}, \sum_{ijk \in \mathcal{V}} \overrightarrow{U_{ijk}}) = 0 \,,$$

from which we have

$$D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{030}}) + D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{003}}) + D^2.(\overrightarrow{U_{030}}, \overrightarrow{U_{003}})$$

$$= -\frac{1}{2} \left\{ D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{300}}) + D^2.(\overrightarrow{U_{030}}, \overrightarrow{U_{030}}) + D^2.(\overrightarrow{U_{003}}, \overrightarrow{U_{003}}) \right\} \,,$$

and Relation (21) becomes

$$\sum_{ijk \in \mathcal{E}} D^2.(\overrightarrow{V_{ijk}}, \overrightarrow{V_{ijk}}) = \frac{2}{3} \left\{ D^2.(\overrightarrow{U_{300}}, \overrightarrow{U_{300}}) + D^2.(\overrightarrow{U_{030}}, \overrightarrow{U_{030}}) + D^2.(\overrightarrow{U_{003}}, \overrightarrow{U_{003}}) \right\} \,,$$

then, identifying this sum in Relations (19) and (20) proves Relation (18) and completes the proof. □

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Since the $p_{ijk}(u, v, w)$s enjoy the same properties, for all the indices ijk, we have

$$12 p_{ijk}(\hat{A}_{111}) + 2 \sum_{lmn \in \mathcal{V}} p_{ijk}(\hat{A}_{lmn}) - 3 \sum_{lmn \in \mathcal{E}} p_{ijk}(\hat{A}_{lmn}) = 0 \,.$$

Now, we use Relation (17) to replace the p_{ijk}s by their counterpart in terms of the complete shape functions. For symmetry reasons, this relation reduces to

$$p_{ijk}(u, v, w) = p^c_{ijk}(u, v, w) + \alpha \, p^c_{111}(u, v, w) \quad \text{for} \quad ijk \in \mathcal{V} \,,$$

$$\text{and} \quad p_{ijk}(u, v, w) = p^c_{ijk}(u, v, w) + \beta \, p^c_{111}(u, v, w) \quad \text{for} \quad ijk \in \mathcal{E} \,,$$

in other words, there is only two coefficients. Let us fix $ijk = 300$, then, we have

$$12 p_{300}(\hat{A}_{111}) + 2 \sum_{lmn \in \mathcal{V}} p_{300}(\hat{A}_{lmn}) - 3 \sum_{lmn \in \mathcal{E}} p_{300}(\hat{A}_{lmn}) = 0 \,,$$

and this resumes to $\alpha = -\frac{1}{6}$, and the same (fix $ijk = 210$) implies that $\beta = \frac{1}{4}$. With these values, we have the reduced shape functions fully defined via (17), see hereafter.

Then the reduced element, seen as a patch reads

$$M(u,v,w) = \sum_{ijk} p_{ijk}(u,v,w) A_{ijk} \,,$$

where ijk lives in $\mathcal{V}$ and $\mathcal{E}$, i.e., **9 indices**. We replace again the p_{ijk}s by means of the p^c_{ijk}s, then we have

$$M(u,v,w) = \sum_{ijk} (\, p^c_{ijk}(u,v,w) + \alpha_{ijk} p^c_{111}(u,v,w) \,)\, A_{ijk} \,,$$

with $\alpha_{ijk} = \alpha$ or β, then this reads also

$$M(u,v,w) = \sum_{ijk} p^c_{ijk}(u,v,w) A_{ijk} + \sum_{ijk} \alpha_{ijk} A_{ijk} p^c_{111}(u,v,w) \,,$$

therefore let

$$A_{111} = \sum_{ijk} \alpha_{ijk} A_{ijk} \,,$$

so that

$$M(u,v,w) = \sum_{ijk} p^c_{ijk}(u,v,w) A_{ijk} \,,$$

with now **10 indices**. In other words, we have *invented* the node A_{111} with which we can define a complete element fully equivalent to the reduced element. As already seen, a complete element is equivalent to the Bézier patch

$$\sum_{i+j+k=3} B^3_{ijk}(u,v,w) P_{ijk} \,,$$

from which we obtain P_{111} and it turns out that P_{111} simply read as

$$P_{111} = \sum_{ijk} \alpha_{ijk} P_{ijk} \,,$$

e.g., the same expression as A_{111}. To prove this, we write A_{111} in two ways, first $A_{111} = \sum_{ijk} \alpha_{ijk} A_{ijk}$ and, second, $A_{111} = \sum_{i+j+k=3} B^3_{ijk}(u,v,w) P_{ijk}$ with $(u,v,w) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, then we have the expression of P_{111} in terms of the P_{ijk} and A_{111}. To complete the proof, in A_{111} we replace the A_{ijk}s by means of the P_{ijk}s.

In practice. The 9-node triangle is easy to analyse. Given its nodes, we find its edge control points by the formulae, here for edge $A_{300}A_{030}$,

$$P_{210} = \frac{-5A_{300} + 18A_{210} - 9A_{120} + 2A_{030}}{6} \quad \text{and} \quad P_{120} = \frac{2A_{300} - 9A_{210} + 18A_{120} - 5A_{030}}{6} \,.$$

Then we compute P_{111} using the above formula. Then, just use Relation (13) to have the control coefficients of the jacobian.

The polynomial space of the reduced element. By construction, this space contains space P^2. Now, because the monomial x^2y is in the complete space, it is easy, by means of instantiations, to see that

$$x^2y = \frac{4}{27}p^c_{021} + \frac{2}{27}p^c_{012} + \frac{1}{27}p^c_{111}\,,$$

then, following Relation (17), we have

$$x^2y = \frac{4}{27}p_{021} + \frac{2}{27}p_{012} + \frac{1}{27}p^c_{111} - \frac{1}{4}\frac{4}{27}p^c_{111} - \frac{1}{4}\frac{2}{27}p^c_{111} = \frac{4}{27}p_{021} + \frac{2}{27}p_{012} - \frac{1}{54}p^c_{111}\,,$$

and the monomial x^2y is **not** in the reduced polynomial space because p^c_{111} still contributes, so it is for xy^2. We repeat the same for x^3. A simple calculation shows that:

$$x^3 = \frac{1}{27}p_{210} + \frac{8}{27}p_{120} + p_{030} + \frac{1}{27}p_{012} + \frac{8}{27}p_{021} + \frac{1}{27}p^c_{111}\,,$$

and the monomial x^3 is **not** in the reduced polynomial space, so it is for y^3. But the combinations $x^3{+}2x^2y$, $x^2y{-}xy^2$ and $y^3{+}2xy^2$ are combinations of the reduced functions (indeed, $\frac{1}{27}+2\frac{-1}{54} = 0$ and $\frac{1}{27} - \frac{1}{27} = 0$). As a conclusion, the diagram of the polynomial space is

$$\begin{array}{ccccc} & & 1 & & \\ & x & & y & \\ x^2 & & xy & & y^2 \\ x^3 + x^2y & & x^2y - xy^2 & & y^3 + xy^2 \end{array}$$

As compared with the polynomial space of the complete element, we miss x^3, y^3, x^2y and xy^2 but we have the above linear combinations of these monomials.

The shape functions. With the value of α and β and using Relation (17), it is easy to obtain the explicit expression of the shape functions. Indeed, only two of them must be made explicit since, for symmetry reasons, the other functions are evident to obtain. Therefore, we use Relation (11) to express $p^c_{300}(u,v), p^c_{210}(u,v)$ and $p^c_{111}(u,v)$ and then compute $p_{300}(u,v)$ and $p_{210}(u,v)$, the two *type* functions, we find

$$p_{300}(u,v,w) = \frac{1}{2}u(2u^2 + 2v^2 + 2w^2 - 5uv - 5uw - 5vw)\,,$$

$$\text{or}\quad p_{300}(x,y) = \frac{9}{2}(1-x-y)(\frac{2}{9} - x - y + xy + x^2 + y^2)\,,$$

$$\text{and}\quad p_{210}(u,v) = \frac{9}{4}uv(4u - 2v + w) \quad \text{or} \quad p_{210}(x,y) = \frac{9}{4}x(1-x-y)(4 - 6x - 3y)\,.$$

The two type functions are:

1	$p_{300}(x,y) = \frac{9}{2}(1-x-y)(\frac{2}{9} - x - y + xy + x^2 + y^2)$
4	$p_{210}(x,y) = \frac{9}{4}x(1-x-y)(4-6x-3y)$

Type shape functions of the reduced 9-node triangle of degree 3

By symmetry $(x \to 1-x, (u,v) \to (v,u), etc., in \quad x \quad or \quad u)$, the full list is the following:

1	$p_{300}(x,y) = \frac{9}{2}(1-x-y)(\frac{2}{9}-x-y+xy+x^2+y^2)$
2	$p_{030}(x,y) = \frac{9}{2}x(\frac{2}{9}-x-y+xy+x^2+y^2)$
3	$p_{003}(x,y) = \frac{9}{2}y(\frac{2}{9}-x-y+xy+x^2+y^2)$
4	$p_{210}(x,y) = \frac{9}{4}x(1-x-y)(4-6x-3y)$
5	$p_{120}(x,y) = \frac{9}{4}x(1-x-y)(-2+6x+3y)$
6	$p_{021}(x,y) = \frac{9}{4}xy(1+3x-3y)$
7	$p_{012}(x,y) = \frac{9}{4}xy(1-3x+3y)$
8	$p_{102}(x,y) = \frac{9}{4}y(1-x-y)(-2+3x+6y)$
9	$p_{201}(x,y) = \frac{9}{4}y(1-x-y)(4-3x-6y)$

Shape functions of the reduced 9-node triangle of degree 3

This concludes the case of the 9-node triangle.

The 12-node triangle of degree 4. To give another example of an higher order reduced triangle, we consider the case $d = 4$. The complete Lagrange triangle has 15 nodes including 3 internal nodes and the numbering of the nodes is as follow:

```
                004                                   004
            103     013                           103     013
        202     112     022         ==>       202             022
    301     211     121     031           301                     031
400     310     220     130     040   400     310     220     130     040
```

The reduced element, if any, has only 12 nodes. Constructing this element involves using 3 Taylor expansions centered at the 3 internal nodes and looking in the reduced space P for space $P^{d-1} = P^3$. From this results two serendipity relations (q being a generic function)

$$15q(\hat{A}_{211}) - 15q(\hat{A}_{121}) = -2q(\hat{A}_{400}) + 2q(\hat{A}_{040})$$

$$-5q(\hat{A}_{130}) - 2q(\hat{A}_{103}) + 5q(\hat{A}_{310}) + 3q(\hat{A}_{202}) + 3q(\hat{A}_{301}) + 2q(\hat{A}_{013}) - 3q(\hat{A}_{031}) - 3q(\hat{A}_{022})\,,$$

and

$$15q(\hat{A}_{211}) - 15q(\hat{A}_{112}) = -2q(\hat{A}_{400}) + 2q(\hat{A}_{004})$$

$$-2q(\hat{A}_{130}) - 5q(\hat{A}_{103}) + 3q(\hat{A}_{310}) + 3q(\hat{A}_{220}) + 5q(\hat{A}_{301}) + 2q(\hat{A}_{031}) - 3q(\hat{A}_{022}) - 3q(\hat{A}_{013})\,. \quad (22)$$

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Proof: We return to that we did for the previous element but, in this case, we consider the three following sub-templates:

```
 .                   .                   004
103   .              . 013               103  013
202  112   .         . 112  022          202  112  022
301  211  121   .    . 211  121  031     301  211  121  031
400  310  220  130 . . 310  220  130  040 .    .    .    .   .
```

and we note $\mathcal{V}_{211}$ and $\mathcal{E}_{211}$. etc., the set of vertices and the set of edge nodes relative to the sub-template centered at $\hat{A}_{211}$, etc. We consider the sub-template depicted in the above diagram (left side) and we use a Taylor expansion to write the value of a function q in $\hat{A}_{400}, ..., \hat{A}_{301}$, the 9 points in the boundary of the sub-template in terms of its value and its derivatives $D^1.(.), D^2.(.,.)$ evaluated at $\hat{A}_{211}$. Since the expansion terminates at order 3, for a vertex, here A_{400}, we have

$$q(\hat{A}_{400}) = q(\hat{A}_{211}) + D^1.(\overrightarrow{U_{400}}) + D^2.(\overrightarrow{U_{400}}, \overrightarrow{U_{400}}) + D^3(\overrightarrow{U_{400}}, \overrightarrow{U_{400}}, \overrightarrow{U_{400}}) \,,$$

where $\overrightarrow{U_{400}} = \overrightarrow{\hat{A}_{111}\hat{A}_{400}}$, the derivatives are evaluated at $\hat{A}_{211}$ and the symbols $D^2.$ and $D^3.$ include (for short) the coefficients, here $\frac{1}{2}$ and $\frac{1}{6}$ The same holds for the other vertices in $\mathcal{V}_{211}$, so, after summation, we have

$$\sum_{ijk\in\mathcal{V}_{211}} q(\hat{A}_{ijk}) = 3q(\hat{A}_{211}) + \sum_{ij\in\mathcal{V}_{211}} D^1.(\overrightarrow{U_{ijk}}) + \sum_{ijk\in\mathcal{V}_{211}} D^2.(\overrightarrow{U_{ijk}}, \overrightarrow{U_{ijk}}) + \sum_{ijk\in\mathcal{V}_{211}} D^3.(\overrightarrow{U_{ijk}}, \overrightarrow{U_{ijk}}, \overrightarrow{U_{ijk}}) \,. \tag{23}$$

A similar relation holds for the nodes in $\mathcal{E}_{211}$ and we have

$$\sum_{ijk\in\mathcal{E}_{211}} q(\hat{A}_{ijk}) = 6q(\hat{A}_{211}) + \sum_{ij\in\mathcal{E}_{211}} D^1.(\overrightarrow{V_{ijk}}) + \sum_{ijk\in\mathcal{E}_{211}} D^2.(\overrightarrow{V_{ijk}}, \overrightarrow{V_{ijk}}) + \sum_{ijk\in\mathcal{E}_{211}} D^3.(\overrightarrow{V_{ijk}}, \overrightarrow{V_{ijk}}, \overrightarrow{V_{ijk}}) \,. \tag{24}$$

where $\overrightarrow{V_{ijk}} = \overrightarrow{\hat{A}_{211}\hat{A}_{ijk}}$ with $ijk \in \mathcal{E}_{211}$.

The derivative D^1 is a linear operator, so $\sum_{ijk\in\mathcal{V}_{211}} D^1.(\overrightarrow{U_{ijk}}) = D^1.(\sum_{ijk\in\mathcal{V}_{211}} \overrightarrow{U_{ijk}}) = 0$, similarly we have$\sum_{ijk\in\mathcal{E}} D^1.(\overrightarrow{V_{ijk}}) = 0$.

On the other hand, $\sum_{ijk\in\mathcal{V}_{211}} D^3.(\overrightarrow{U_{ijk}}, \overrightarrow{U_{ijk}}, \overrightarrow{U_{ijk}}) \neq 0$ while $\sum_{ijk\in\mathcal{E}_{211}} D^3.(\overrightarrow{V_{ijk}}, \overrightarrow{V_{ijk}}, \overrightarrow{V_{ijk}}) = 0$ and the two above relations simplify.

Now, $\overrightarrow{V_{ijk}}$ can be written in terms of the $\overrightarrow{U_{ijk}}$. Indeed, let us consider $\hat{A}_{310}$, we have $\hat{A}_{310} = \frac{2\hat{A}_{400}+\hat{A}_{130}}{3}$ so $\overrightarrow{V_{310}} = \frac{\overrightarrow{2U_{400}}+\overrightarrow{U_{130}}}{3}$ holds. Since, again for index$_{310}$, we have

$$D^2.(\overrightarrow{V_{310}}, \overrightarrow{V_{310}}) = D^2.(\frac{\overrightarrow{2U_{400}} + \overrightarrow{U_{130}}}{3}, \frac{\overrightarrow{2U_{400}} + \overrightarrow{U_{130}}}{3})$$

$$= \frac{4}{9}D^2.(\overrightarrow{U_{400}}, \overrightarrow{U_{400}}) + \frac{4}{9}D^2.(\overrightarrow{U_{400}}, \overrightarrow{U_{130}}) + \frac{1}{9}D^2.(\overrightarrow{U_{130}}, \overrightarrow{U_{130}}) \,,$$

and, by symmetry

$$D^2.(\overrightarrow{V_{220}}, \overrightarrow{V_{220}}) = \frac{1}{9}D^2.(\overrightarrow{U_{400}}, \overrightarrow{U_{400}}) + \frac{4}{9}D^2.(\overrightarrow{U_{400}}, \overrightarrow{U_{130}}) + \frac{4}{9}D^2.(\overrightarrow{U_{130}}, \overrightarrow{U_{130}}) \,,$$

and similar expressions for the other egde nodes, we have

$$\sum_{ijk\in\mathcal{E}_{211}} D^2.(\overrightarrow{V_{ijk}}, \overrightarrow{V_{ijk}}) = \frac{10}{9}D^2.(\overrightarrow{U_{400}}, \overrightarrow{U_{400}}) + \frac{10}{9}D^2.(\overrightarrow{U_{130}}, \overrightarrow{U_{130}}) + \frac{10}{9}D^2.(\overrightarrow{U_{103}}, \overrightarrow{U_{103}})$$

$$+ \frac{8}{9}D^2.(\overrightarrow{U_{400}}, \overrightarrow{U_{130}}) + \frac{8}{9}D^2.(\overrightarrow{U_{400}}, \overrightarrow{U_{103}}) + \frac{8}{9}D^2.(\overrightarrow{U_{130}}, \overrightarrow{U_{103}}) \,, \tag{25}$$

in other words, the sum for ijk in $\mathcal{E}_{211}$ is replaced by sums for ijk in $\mathcal{V}_{211}$. Now, because $\sum_{ijk\in\mathcal{V}_{211}} \overrightarrow{U_{ijk}} = 0$, we have

$$D^2.(\sum_{ijk\in\mathcal{V}_{211}} \overrightarrow{U_{ijk}}, \sum_{ijk\in\mathcal{V}_{211}} \overrightarrow{U_{ijk}}) = 0 \,,$$

from which we have

$$D^2.(\overrightarrow{U_{400}},\overrightarrow{U_{130}}) + D^2.(\overrightarrow{U_{400}},\overrightarrow{U_{103}}) + D^2.(\overrightarrow{U_{130}},\overrightarrow{U_{103}})$$
$$= -\frac{1}{2}\left\{D^2.(\overrightarrow{U_{400}},\overrightarrow{U_{400}}) + D^2.(\overrightarrow{U_{130}},\overrightarrow{U_{130}}) + D^2.(\overrightarrow{U_{103}},\overrightarrow{U_{103}})\right\},$$

and Relation (25) becomes

$$\sum_{ijk\in\mathcal{E}_{211}} D^2.(\overrightarrow{V_{ijk}},\overrightarrow{V_{ijk}}) = \frac{2}{3}\sum_{ijk\in\mathcal{V}_{211}} D^2.(\overrightarrow{U_{ijk}},\overrightarrow{U_{ijk}}),,$$

and, now, Relations (19) and (20) respectively reduce to

$$\sum_{ijk\in\mathcal{V}_{211}} q(\hat{A}_{ijk}) = 3q(\hat{A}_{211}) + \sum_{ijk\in\mathcal{V}_{211}} D^2.(\overrightarrow{U_{ijk}},\overrightarrow{U_{ijk}}) + \sum_{ijk\in\mathcal{V}_{211}} D^3.(\overrightarrow{U_{ijk}},\overrightarrow{U_{ijk}},\overrightarrow{U_{ijk}}),$$

and

$$\sum_{ijk\in\mathcal{E}_{211}} q(\hat{A}_{ijk}) = 6q(\hat{A}_{211}) + \sum_{ijk\in\mathcal{E}_{211}} D^2.(\overrightarrow{V_{ijk}},\overrightarrow{V_{ijk}}) \;=\; 6q(\hat{A}_{211}) + \frac{2}{3}\sum_{ijk\in\mathcal{V}_{211}} D^2.(\overrightarrow{U_{ijk}},\overrightarrow{U_{ijk}}).$$

A linear combination of these two lines leads to

$$2\sum_{ijk\in\mathcal{V}_{211}} q(\hat{A}_{ijk}) \;-\; 3\sum_{ijk\in\mathcal{E}_{211}} q(\hat{A}_{ijk}) \;=\; -12\,q(\hat{A}_{211}) \;+\; 2\sum_{ijk\in\mathcal{V}_{211}} D^3.(\overrightarrow{U_{ijk}},\overrightarrow{U_{ijk}},\overrightarrow{U_{ijk}}),$$

which is written as

$$2\sum_{ijk\in\mathcal{V}_{211}} D^3.(\overrightarrow{U_{ijk}},\overrightarrow{U_{ijk}},\overrightarrow{U_{ijk}}) \;=\; 12\,q(\hat{A}_{211}) \;+\; 2\sum_{ijk\in\mathcal{V}_{211}} q(\hat{A}_{ijk}) \;-\; 3\sum_{ijk\in\mathcal{E}_{211}} q(\hat{A}_{ijk}).$$

The $2\,D^3.(.,.,.)$ is a constant denoted as C_{211}, so we have

$$C_{211} \;=\; 12\,q(\hat{A}_{211}) \;+\; 2\sum_{ijk\in\mathcal{V}_{211}} q(\hat{A}_{ijk}) \;-\; 3\sum_{ijk\in\mathcal{E}_{211}} q(\hat{A}_{ijk}), \tag{26}$$

and the same applies to the two other sub-templates, so we also have

$$C_{121} \;=\; 12\,q(\hat{A}_{211}) \;+\; 2\sum_{ijk\in\mathcal{V}_{121}} q(\hat{A}_{ijk}) \;-\; 3\sum_{ijk\in\mathcal{E}_{121}} q(\hat{A}_{ijk}),$$

$$C_{112} \;=\; 12\,q(\hat{A}_{112}) \;+\; 2\sum_{ijk\in\mathcal{V}_{112}} q(\hat{A}_{ijk}) \;-\; 3\sum_{ijk\in\mathcal{E}_{112}} q(\hat{A}_{ijk}).$$

We write $C_{211} = C_{121} = C_{112}$ and the Relations (22) hold, therefore completing the proof. □

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Since the $p_{ijk}(u,v,w)$s enjoy the same properties, for all the indices ijk, Relations (22) also hold for these functions. Now, we use Relation (17) to replace the p_{ijk}s by their counterpart in terms of the complete shape functions. For short, we write this relation as

$$p_{ijk}(u,v,w) = p^c_{ijk}(u,v,w) + \alpha_{ijk}p^c_{211}(u,v,w) + \beta_{ijk}p^c_{121}(u,v,w) + \gamma_{ijk}p^c_{112}(u,v,w).$$

Due to the symmetries, only three indices must be examined, $_{400}$, $_{310}$ and $_{220}$. For index $_{400}$, Relations (22) simply reduce to

$$15\alpha_{400} - 15\beta_{400} = -2 \quad \text{and} \quad 15\alpha_{400} - 15\gamma_{400} = -2\,,$$

therefore, we have only one parameter, say α_1, and we have

$$\alpha_{400} = \alpha_1,\ \beta_{400} = \alpha_1 + \frac{2}{15}\,\text{and}\,\gamma_{400} = \beta_{400} = \alpha_1 + \frac{2}{15}\,.$$

For index $_{310}$, we have

$$15\alpha_{310} - 15\beta_{310} = 5\,\text{or}\,3\alpha_{310} - 3\beta_{310} = 1 \quad \text{and} \quad 15\alpha_{310} - 15\gamma_{310} = 3\,\text{or}\,5\alpha_{310} - 5\gamma_{310} = 1\,,$$

therefore with one parameter, say α_2, and we have

$$\alpha_{310} = \alpha_2,\ \beta_{310} = \alpha_2 - \frac{1}{3}\,\text{and}\,\gamma_{310} = \alpha_2 - \frac{1}{5}\,.$$

For index $_{220}$, we have

$$15\alpha_{220} - 15\beta_{220} = 0 \quad \text{and} \quad 15\alpha_{220} - 15\gamma_{220} = 3\,\text{or}\,5\alpha_{220} - 5\gamma_{220} = 1\,,$$

therefore with one parameter, say α_3, and we have

$$\alpha_{220} = \alpha_3,\ \beta_{220} = \alpha_3\,\text{and}\,\gamma_{220} = \alpha_3 - \frac{1}{5}\,.$$

Again by symmetries, we have all the coefficients in terms of the three introduced parameters, see the following table:

i	ijk	α_{ijk}	β_{ijk}	γ_{ijk}
1	400	α_1	$\alpha_1 + 2/15$	$\alpha_1 + 2/15$
2	040	$\alpha_1 + 2/15$	α_1	$\alpha_1 + 2/15$
3	004	$\alpha_1 + 2/15$	$\alpha_1 + 2/15$	α_1
4	310	α_2	$\alpha_2 - 1/3$	$\alpha_2 - 1/5$
7	031	$\alpha_2 - 1/5$	α_2	$\alpha_2 - 1/3$
10	103	$\alpha_2 - 1/3$	$\alpha_2 - 1/5$	α_2
6	130	$\alpha_2 - 1/3$	α_2	$\alpha_2 - 1/5$
9	013	$\alpha_2 - 1/5$	$\alpha_2 - 1/3$	α_2
12	301	α_2	$\alpha_2 - 1/5$	$\alpha_2 - 1/3$
5	220	α_3	α_3	$\alpha_3 - 1/5$
8	022	$\alpha_3 - 1/5$	α_3	α_3
11	202	α_3	$\alpha_3 - 1/5$	α_3

Coefficients for the repartition in terms of the three parameters.

Since $\sum_{ijk} \alpha_{ijk} = \sum_{ijk} \beta_{ijk} = \sum_{ijk} \gamma_{ijk} = 1$, we find the relation

$$3\alpha_1 + 6\alpha_2 + 3\alpha_3 = 2\,. \tag{27}$$

As a consequence and *a priori* , we have as solution a family of triangles defined by two parameters. To make precise a choice, the method is to find some additional relations between the two parameters by imposing some additional monomials (or linear combinations of monomials) in the polynomial space of the reduced element. By construction, we have the space P^3 included in the reduced polynomial space. As an exercice, we check that x is effectively present. By instantiations, we know that x is in the complete space and (we evidently write the shape functions in terms of (x, y) and not in terms of (u, v, w)) reads

$$x = \frac{1}{4}p^c_{310}(x,y) + \frac{2}{4}p^c_{220}(x,y) + \frac{3}{4}p^c_{130}(x,y) + p^c_{040}(x,y) + \frac{3}{4}p^c_{031}(x,y) + \frac{2}{4}p^c_{022}(x,y) + \frac{1}{4}p^c_{013}(x,y)$$

$$+\frac{1}{4}p^c_{211}(x,y) + \frac{2}{4}p^c_{121}(x,y) + \frac{1}{4}p^c_{112}(x,y)\,,$$

but we have, here for index $_{310}$,

$$p_{310}(x,y) = p^c_{310}(x,y) + \alpha_{310}p^c_{211}(x,y) + \beta_{310}p^c_{121}(x,y) + \gamma_{310}p^c_{2112}(x,y)$$

then, conversely, we have

$$p^c_{310}(x,y) = p_{310}(x,y) - \alpha_{310}p^c_{211}(x,y) - \beta_{310}p^c_{121}(x,y) - \gamma_{310}p^c_{2112}(x,y)\,,$$

together with similar expressions for all the functions involves in x. Now, we replace the complete functions by means of these expressions and, looking only for the terms in p^c_{211}, we have

$$x = \frac{1}{4}(p_{310}(x,y) - \alpha_{310}p^c_{211}(x,y)) + \frac{2}{4}(p_{220}(x,y) - \alpha_{220}p^c_{211}(x,y)) + \frac{3}{4}(p_{130}(x,y) - \alpha_{130}p^c_{211}(x,y))$$

$$+(p_{040}(x,y) - \alpha_{040}p^c_{211}(x,y)) + \frac{3}{4}(p_{031}(x,y) - \alpha_{031}p^c_{211}(x,y)) + \frac{2}{4}(p_{022}(x,y) - \alpha_{022}p^c_{211}(x,y))$$

$$+\frac{1}{4}(p_{013}(x,y) - \alpha_{013}p^c_{211}(x,y)) + \frac{1}{4}p^c_{211}(x,y) + \ldots\,,$$

so we have

$$x = \frac{1}{4}p_{310}(x,y) + \frac{2}{4}p_{220}(x,y) + \frac{3}{4}p_{130}(x,y) + p_{040}(x,y) + \frac{3}{4}p_{031}(x,y) + \frac{2}{4}p_{022}(x,y) + \frac{1}{4}p_{013}(x,y)$$

$$+\left\{-\frac{1}{4}\alpha_{310} - \frac{2}{4}\alpha_{220} - \frac{3}{4}\alpha_{130} - \alpha_{040} - \frac{3}{4}\alpha_{031} - \frac{2}{4}\alpha_{022} - \frac{1}{4}\alpha_{013} + \frac{1}{4}\right\} p^c_{211}(x,y) \;+\; \ldots\,,$$

and we write the α_{ijk}s in terms of the above three parameters, then, the coefficient in $p^c_{211}(x, y)$ reduces to

$$-\alpha_1 - 2\alpha_2 - \alpha_3 + \frac{2}{3}\,,$$

thus, via the Relation (27), the contribution of $p^c_{211}(x, y)$ is null, so it is for the two other "central" functions, therefore, with no surprise, x is in the reduced polynomial space.

Before going further, we consider the monomial x^2. From x and more precisely from $4x$, e.g., from the relation

$$4x = p^c_{310}(x,y) + 2p^c_{220}(x,y) + 3p^c_{130}(x,y) + 4p^c_{040}(x,y) + 3p^c_{031}(x,y) + 2p^c_{022}(x,y) + p^c_{013}(x,y)$$

$$+p^c_{211}(x,y) + 2p^c_{121}(x,y) + p^c_{112}(x,y)\,,$$

we mechanically obtain

$$16x^2 = p^c_{310}(x,y) + 4p^c_{220}(x,y) + 9p^c_{130}(x,y) + 16p^c_{040}(x,y) + 9p^c_{031}(x,y) + 4p^c_{022}(x,y)$$

$$+p^c_{013}(x,y)+p^c_{211}(x,y)+4p^c_{121}(x,y)+p^c_{112}(x,y)\,,$$

then, as above, we replace the p^c_{ijk}s by the corresponding p_{ijk}s and the adequate repartition of the "central" functions. Then, as regard to $p^c_{211}(x,y)$, we find the relation

$$16x^2=(p_{310}(x,y)-\alpha_{310}p^c_{211}(x,y))+4(p_{220}(x,y)-\alpha_{220}p^c_{211}(x,y))+9(p_{130}(x,y)-\alpha_{130}p^c_{211}(x,y))$$

$$+(p_{040}(x,y)-\alpha_{040}p^c_{211}(x,y))+9(p_{031}(x,y)-\alpha_{031}p^c_{211}(x,y))+4(p_{022}(x,y)-\alpha_{022}p^c_{211}(x,y))$$

$$(p_{013}(x,y)-\alpha_{013}p^c_{211}(x,y))+p^c_{211}(x,y)+\ldots\,,$$

and it yields the coefficient (omitting the factor 16)

$$-(\alpha_{310}+4\alpha_{220}+9\alpha_{130}+16\alpha_{040}+9\alpha_{031}+4\alpha_{022}+\alpha_{013})+1\,,$$

where we write the α_{ijk}s in terms of α_1,α_2 and α_3, then we have the relation

$$0=-(\alpha_2+4\alpha_3+9(\alpha_2-1/3)+16(\alpha_1+2/15)+9(\alpha_2-1/5)+4(\alpha_3-1/5)+(\alpha_2-1/5)\,)+1$$

from which we obtain the relation

$$-16\alpha_1-20\alpha_2-8\alpha_3=-14/3\,,$$

which gives the equation

$$8\alpha_1+10\alpha_2+4\alpha_3=7/3\,. \tag{28}$$

Now, we turn to xy. Since:

$$16xy=3p^c_{031}(x,y)+4p^c_{022}(x,y)$$
$$+3p^c_{013}(x,y)+p^c_{211}(x,y)+2p^c_{121}(x,y)+2p^c_{112}(x,y)\,,$$

as regard to $p^c_{211}(x,y)$, we find

$$16xy=3(p_{031}(x,y)-\alpha_{031}p^c_{211}(x,y))+4(p_{022}(x,y)-\alpha_{022}p^c_{211}(x,y))$$

$$+3(p_{013}(x,y)-\alpha_{013}p^c_{211}(x,y))+p^c_{211}(x,y)+\ldots\,,$$

and we have the coefficient

$$-3\alpha_{031}-4\alpha_{022}-3\alpha_{013}+1\;=\;-3(\alpha_2-1/5)-4(\alpha_3-1/5)-3(\alpha_2-1/5)+1\,,$$

which gives the equation

$$6\alpha_2+4\alpha_3=3\,. \tag{29}$$

Now we group together the 3 equations (27), (28) and (29) and the resulting system has no solution.

As a conclusion, there is no way to find a reduced element covering P^3. The sole chance to find a solution is to reduce our quest by imposing only the P^2 space. To this end we return to the three relations

$$C_{211}\;=\;12\,q(\hat A_{211})\;+\;2\sum_{ijk\in\mathcal{V}_{211}}q(\hat A_{ijk})\;-\;3\sum_{ijk\in\mathcal{E}_{211}}q(\hat A_{ijk})\,,$$

$$C_{121}\;=\;12\,q(\hat A_{211})\;+\;2\sum_{ijk\in\mathcal{V}_{121}}q(\hat A_{ijk})\;-\;3\sum_{ijk\in\mathcal{E}_{121}}q(\hat A_{ijk})\,,$$

$$C_{112} \;=\; 12\, q(\hat{A}_{112}) \,+\, 2 \sum_{ijk\in\mathcal{V}_{112}} q(\hat{A}_{ijk}) \,-\, 3 \sum_{ijk\in\mathcal{E}_{112}} q(\hat{A}_{ijk})\,,$$

with the (restricted) constraints $C_{211} = 0, C_{121} = 0$ and $C_{112} = 0$. This results to the following system:

$$\begin{aligned}
12q(\hat{A}_{211}) - 3q(\hat{A}_{121}) - 3q(\hat{A}_{112}) &= -2q(\hat{A}_{400}) - 2q(\hat{A}_{130}) - 2q(\hat{A}_{103}) \\
&\quad +3q(\hat{A}_{310}) + 3q(\hat{A}_{220}) + 3q(\hat{A}_{202}) + 3q(\hat{A}_{301})\,, \\
-3q(\hat{A}_{211}) + 12q(\hat{A}_{121}) - 3q(\hat{A}_{112}) &= -2q(\hat{A}_{310}) - 2q(\hat{A}_{040}) - 2q(\hat{A}_{013}) \\
&\quad +3q(\hat{A}_{220}) + 3q(\hat{A}_{130}) + 3q(\hat{A}_{031}) + 3q(\hat{A}_{022})\,, \\
-3q(\hat{A}_{211}) - 3q(\hat{A}_{121}) + 12q(\hat{A}_{112}) &= -2q(\hat{A}_{301}) - 2q(\hat{A}_{031}) - 2q(\hat{A}_{004}) \\
&\quad +3q(\hat{A}_{022}) + 3q(\hat{A}_{013}) + 3q(\hat{A}_{103}) + 3q(\hat{A}_{202})\,.
\end{aligned}$$

Now, we replace the generic function q by the p_{ijk}s and we use Relation (17) to replace these p_{ijk}s by their counterpart in terms of the complete shape functions. We write

$$p_{ijk}(u,v,w) = p^c_{ijk}(u,v,w) + \alpha_{ijk}\, p^c_{211}(u,v,w) + \beta_{ijk}\, p^c_{121}(u,v,w) + \gamma_{ijk}\, p^c_{112}(u,v,w)\,,$$

and we consider the three *type* functions, e.g. for $ijk = 400, ijk = 310$ and $ijk = 220$. The previous system gives the corresponding coefficients and, by symmetries, all the coefficients are obtained as reported in the following table

i	ijk	α_{ijk}	β_{ijk}	γ_{ijk}
1	400	$-1/5$	$-1/15$	$-1/15$
2	040	$-1/15$	$-1/5$	$-1/15$
3	004	$-1/15$	$-1/15$	$-1/5$
4	310	$7/30$	$-1/10$	$1/30$
7	031	$1/30$	$7/30$	$-1/10$
10	103	$-1/10$	$1/30$	$7/30$
6	130	$-1/10$	$7/30$	$1/30$
9	013	$1/30$	$-1/10$	$7/30$
12	301	$7/30$	$1/30$	$-1/10$
5	220	$2/5$	$2/5$	$1/5$
8	022	$1/5$	$2/5$	$2/5$
11	202	$2/5$	$1/5$	$2/5$

Coefficients for the repartition of the 12-node triangle.

and the following diagram displays the coefficients for the repartition corresponding to $\hat{A}_{211}$.

```
-1/15

-1/10  1/30

 2/5         1/5

 7/30  [211]       1/30

-1/5   7/30  2/5  -1/10  -1/15
```

From this, see herefater, we can compute the reduced shape functions.

Then, with the reduced shape functions, the reduced element, seen as a patch, reads

$$M(u,v,w)=\sum_{ijk} p_{ijk}(u,v,w)A_{ijk}\,,$$

where the ijks consist in **12 indices**. We replace again the p_{ijk}s by means of the p^c_{ijk}s, then we have

$$M(u,v,w)=\sum_{ijk}(\,p^c_{ijk}(u,v,w)+\alpha_{ijk}p^c_{211}(u,v,w)+\beta_{ijk}p^c_{121}(u,v,w)+\gamma_{ijk}p^c_{112}(u,v,w)\,)\,A_{ijk}\,,$$

with the coefficients of the above table, then this reads also

$$M(u,v,w)=\sum_{ijk} p^c_{ijk}(u,v,w)A_{ijk}+\sum_{ijk}\alpha_{ijk}A_{ijk}p^c_{211}(u,v,w)$$

$$+\sum_{ijk}\beta_{ijk}A_{ijk}p^c_{121}(u,v,w)+\sum_{ijk}\gamma_{ijk}A_{ijk}p^c_{112}(u,v,w)\,,$$

therefore let

$$A_{211}=\sum_{ijk}\alpha_{ijk}A_{ijk}\,,$$

with **12 indices**, and similar expressions for A_{121} and A_{112}, so that

$$M(u,v,w)=\sum_{ijk} p^c_{ijk}(u,v,w)A_{ijk}\,,$$

with now **15 indices**. In other words, we have *invented* the nodes A_{211}, A_{121} and A_{112} with which we can define a complete element fully equivalent to the reduced element. As already seen, a complete element is equivalent to the Bézier patch

$$\sum_{i+j+k=4} B^4_{ijk}(u,v,w)P_{ijk}\,,$$

from which we obtain P_{211}, P_{121} and P_{112}. One can intuitively think that $P_{211}, ...$ simply reads as

$$P_{211}=\sum_{ijk}\alpha_{ijk}P_{ijk}\,, ...\,,$$

e.g., the same expression as $A_{211}, ...$, but, see [19], it is wrong.

In practice. The 12-node triangle analyse, given its nodes, leads to finding its edge control points by the formulae, here for edge $A_{400}A_{040}$,

$$P_{310}=\frac{-13A_{400}+48A_{310}-36A_{220}+16A_{130}-3A_{040}}{12}\,,$$

$$P_{220}=\frac{13A_{400}-64A_{310}+120A_{220}-64A_{130}+13A_{040}}{18}\,,$$

$$P_{130}=\frac{-3A_{400}+16A_{310}-36A_{220}+48A_{130}-13A_{040}}{12}\,.$$

Then we compute P_{211}, P_{121} and P_{112} by solving the 3×3 corresponding system. Then, just use Relation (13) to have the control coefficients of the jacobian.

The polynomial space of the reduced element. As seen during the construction of the solution, the polynomial space contains P^2 and various instantiations show that some other linear combinations of monomials are covered as depicted in the following diagram

$$\begin{array}{cccc} & & 1 & & \\ & x & & y & \\ & x^2 & xy & y^2 & \\ x^3+2x^2y & x^3+2xy^2 & y^3+2x^2y & y^3+2xy^2 \end{array}$$

and, therefore, x^3-y^3 together with x^2y-xy^2 are also covered.

The shape functions. As previously seen, it is only required to write the three *type* functions, p_{400}, p_{310} and p_{220} to have all of them by symmetries. To this end we need to explicit p^c_{400}, p^c_{310}, p^c_{220} together with p^c_{211}, p^c_{121}, p^c_{112}, the "central" functions. After Relation (11), we have

$$p^c_{400}(u,v,w) = \frac{1}{6}u(4u-1)(4u-2)(4u-3)\,,$$

$$p^c_{310}(u,v,w) = \frac{8}{3}u(4u-1)(4u-2)v\,,$$

$$p^c_{220}(u,v,w) = 4u(4u-1)v(4v-1)\,,$$

$p^c_{211}(u,v,w) = 32uvw(4u-1)\,, p^c_{121}(u,v,w) = 32uvw(4v-1)$ and $p^c_{112}(u,v,w) = 32uvw(4w-1)\,.$

Then, for index $_{400}$, we have

$$p_{400}(u,v,w) = \frac{1}{6}u(4u-1)(4u-2)(4u-3) - \frac{32}{15}uvw(3(4u-1)+(4v-1)+(4w-1))\,,$$

then, using `Maple`, we have

$$(32/3)u^4 - 16u^3 + (22/3)u^2 - u - (128/5)u^2vw + (32/3)uvw - (128/15)uv^2w - (128/15)uvw^2\,,$$

and, in terms of x and y, this reads

$$\frac{1}{15}(1-x-y)(15-110x-110y+240x^2+256xy+240y^2-160x^3+224x^2y+224xy^2-160y^3)\,.$$

For index $_{310}$, we obtain

$$p_{310}(u,v,w) = \frac{8}{3}u(4u-1)(4u-2)v + \frac{32}{30}uvw(7(4u-1)-3(4v-1)+(4w-1))\,,$$

then, using `Maple`, we have

$$(128/3)u^3v - 32u^2v + (16/3)uv + (448/15)u^2vw - (16/3)uvw - (64/5)uv^2w + (64/15)uvw^2)\,,$$

and, in terms of x and y, this reads

$$\frac{16}{15}x(1-x-y)(15-50x-27y+40x^2+16y^2+40xy)\,.$$

For index $_{220}$, we obtain

$$p_{220}(u,v,w) = 4u(4u-1)v(4v-1) + \frac{32}{5}uvw(2(4u-1)+2(4v-1))+(4w-1))\,,$$

then, using `Maple`, we have

$$64u^2v^2 - 16u^2v - 16uv^2 + 4uv + (256/5)u^2vw - 32uvw + (256/5)uv^2w + (128/5)uvw^2\,,$$

and, in terms of x and y, this reads

$$\frac{4}{5}x(1-x-y)(-15+80x+44y-80x^2-32y^2-80xy)\,.$$

The following table reports these functions:

1	$p_{400}(x,y) = \frac{1}{15}(1-x-y)$ $(15-110x-110y+240x^2+256xy+240y^2-160x^3+224x^2y+224xy^2-160y^3)$
4	$p_{310}(x,y) = \frac{16}{15}x(1-x-y)(15-50x-27y+40x^2+16y^2+40xy)$
5	$p_{220}(x,y) = \frac{4}{5}x(1-x-y)(-15+80x+44y-80x^2-32y^2-80xy)$

Type shape functions of the reduced 12-node triangle of degree 4.

To have all the shape functions, we simply apply adequate symmetries and rotations to the three type functions using the form in (u,v,w) before writting the solution in (x,y). As an example, one obtains the other functions of edge $400-040$ by permuting u and v. Then, for edge $400-004$, the functions are those of edge $400-040$ by changing v by w. To end, the functions of edge $040-004$ are obtained from those of edge $400-004$ by permuting u and v.

This concludes the case of the 12-node triangle.

Reduced triangles of higher degree? Is there any higher order (e.g., $5,6$, etc.) reduced triangles, actually we don't thing so apart if only the P^2 space is imposed and discussing this point is necessarily a rather technical task, therefore, we don't pursuit this story of reduced triangles.

5.2 Order d serendipity quadrilaterals

The 8-node quad of degree 2. For $d=2$, we have the 9-node quad with one internal node and the numbering of the nodes is as follow:

```
02  12  22         02  12  22
01  11  21   ==>   01      21
00  10  20         00  10  20
```

To define the reduced element, we impose space $P^d = P^2$ to be included in the reduced polynomial space. Then, the Taylor expansion gives the serendipity relation

$$4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{V}} q(\hat{A}_{ij}) - 2\sum_{ij\in\mathcal{E}} q(\hat{A}_{ij}) = 0\,, \tag{30}$$

where $\hat{A}_{11}$ is the point, in the reference element $(\hat{K})$, of coordinates $(\frac{1}{2},\frac{1}{2})$, $\mathcal{V}$ is the set of the vertices in $\hat{K}$ and $\mathcal{E}$ is the set of the edge nodes in $\hat{K}$.

This Serendipity relation also insures that the monomials u^2v and uv^2 are part of the reduced polynomial space.

$\star$

$\star \quad \star$

Proof: We consider the template depicted in the above diagram (left side) and we use a Taylor expansion to write the value of a function q in $\hat{A}_{00}, ..., \hat{A}_{01}$, the 8 points in the boundary of the template in terms of its value and its derivatives $D^1.(.), D^2.(.,.)$ evaluated at $\hat{A}_{11}$. Since the expansion terminates at order 2, for a vertex, here A_{00}, we have

$$q(\hat{A}_{00}) = q(\hat{A}_{11}) + D^1.(\overrightarrow{U_{00}}) + D^2.(\overrightarrow{U_{00}}, \overrightarrow{U_{00}}) \,,$$

where $\overrightarrow{U_{00}} = \overrightarrow{\hat{A}_{11}\hat{A}_{00}}$, the derivatives are evaluated at $\hat{A}_{11}$ and the symbol $D^2.$ includes (for short) the coefficient, here $\frac{1}{2}$. The same holds for the other vertices in $\mathcal{V}$, so, after summation, we have

$$\sum_{ij\in\mathcal{V}} q(\hat{A}_{ij}) = 4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{V}} D^1.(\overrightarrow{U_{ij}}) + \sum_{ij\in\mathcal{V}} D^2.(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}) \,. \tag{31}$$

A similar relation holds for the four nodes in $\mathcal{E}$ and, after summation, we have

$$\sum_{ij\in\mathcal{E}} q(\hat{A}_{ij}) = 4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{E}} D^1.(\overrightarrow{V_{ij}}) + \sum_{ij\in\mathcal{E}} D^2.(\overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}) \,. \tag{32}$$

where $\overrightarrow{V_{ij}} = \overrightarrow{\hat{A}_{11}\hat{A}_{ij}}$ with $ij \in \mathcal{E}$.

The derivative D^1 is a linear operator, so $\sum_{ij\in\mathcal{V}} D^1.(\overrightarrow{U_{ij}}) = D^1.(\sum_{ij\in\mathcal{V}} \overrightarrow{U_{ij}}) = 0$, similarly we have$\sum_{ij\in\mathcal{E}} D^1.(\overrightarrow{V_{ij}}) = 0$. Now, $\overrightarrow{V_{ij}}$ can be written in terms of the $\overrightarrow{U_{ij}}$. Indeed, let us consider $\hat{A}_{10}$, we have $\hat{A}_{10} = \frac{\hat{A}_{00}+\hat{A}_{20}}{2}$ so $\overrightarrow{V_{10}} = \frac{\overrightarrow{U_{00}}+\overrightarrow{U_{20}}}{2}$ holds. Since, again for index$_{10}$, we have

$$D^2.(\overrightarrow{V_{10}}, \overrightarrow{V_{10}}) = D^2.(\frac{\overrightarrow{U_{00}}+\overrightarrow{U_{20}}}{2}, \frac{\overrightarrow{U_{00}}+\overrightarrow{U_{20}}}{2}) = \frac{1}{4}D^2.(\overrightarrow{U_{00}}, \overrightarrow{U_{00}}) + \frac{1}{2}D^2.(\overrightarrow{U_{00}}, \overrightarrow{U_{20}}) + \frac{1}{4}D^2.(\overrightarrow{U_{20}}, \overrightarrow{U_{20}}) \,,$$

and similar expressions for the other indices, we have

$$\sum_{ij\in\mathcal{E}} D^2.(\overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}) = \frac{1}{2}\sum_{ij\in\mathcal{V}} D^2.(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}})$$

$$+\frac{1}{2}\left\{D^2.(\overrightarrow{U_{00}}, \overrightarrow{U_{20}}) + D^2.(\overrightarrow{U_{20}}, \overrightarrow{U_{22}}) + D^2.(\overrightarrow{U_{22}}, \overrightarrow{U_{02}}) + D^2.(\overrightarrow{U_{02}}, \overrightarrow{U_{00}})\right\} \,,$$

but this last sum is null. Indeed,

$$D^2.(\overrightarrow{U_{00}}, \overrightarrow{U_{20}}) + D^2.(\overrightarrow{U_{20}}, \overrightarrow{U_{22}}) + D^2.(\overrightarrow{U_{22}}, \overrightarrow{U_{02}}) + D^2.(\overrightarrow{U_{02}}, \overrightarrow{U_{00}}) = 0 \,.$$

and, for the first two terms we have

$$D^2.(\overrightarrow{U_{00}}, \overrightarrow{U_{20}})+D^2.(\overrightarrow{U_{20}}, \overrightarrow{U_{22}}) = D^2.(\overrightarrow{U_{00}}, \overrightarrow{U_{20}})+D^2.(\overrightarrow{U_{22}}, \overrightarrow{U_{20}}) = D^2.(\overrightarrow{U_{00}}+\overrightarrow{U_{22}}, \overrightarrow{U_{20}}) = D^2.(\overrightarrow{0}, \overrightarrow{U_{20}}) \,,$$

while for the two others, we have

$$D^2.(\overrightarrow{U_{22}}, \overrightarrow{U_{02}})+D^2.(\overrightarrow{U_{02}}, \overrightarrow{U_{00}}) = D^2.(\overrightarrow{U_{22}}, \overrightarrow{U_{02}})+D^2.(\overrightarrow{U_{00}}, \overrightarrow{U_{02}}) = D^2.(\overrightarrow{U_{22}}+\overrightarrow{U_{00}}, \overrightarrow{U_{02}}) = D^2.(\overrightarrow{0}, \overrightarrow{U_{02}}) \,,$$

and the sum is

$$D^2.(\overrightarrow{0}, \overrightarrow{U_{20}}) + D^2.(\overrightarrow{0}, \overrightarrow{U_{02}}) = D^2.(\overrightarrow{0}, \overrightarrow{U_{20}} + \overrightarrow{U_{02}}) = D^2.(\overrightarrow{0}, \overrightarrow{0}) = 0 \,.$$

Therefore

$$\sum_{ij\in\mathcal{E}} D^2.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) = \frac{1}{2}\sum_{ij\in\mathcal{V}} D^2.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})\,,$$

and identifying this sum in Relations (31) and (32) proves Relation (30) and completes the proof.

For a further usage, we introduce the value C_{11} defined by

$$C_{11} = \sum_{ij\in\mathcal{E}} D^2.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}} D^2.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})\,,$$

and the condition $C_{11}=0$ is the *constitutive* definition of the reduced element from which results Relation (30).

This constitutive condition holds for the monomials u^2v and uv^2. Indeed, for these monomails, we consider a Taylor expansion of order 3 and, in this case, C_{11} is written as

$$C_{11} = \sum_{ij\in\mathcal{E}} D^2.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}} D^2.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) + \sum_{ij\in\mathcal{E}} D^3.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}} D^3.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})\,.$$

But similarly to the D^1 derivatives (and for the same reason) we have

$$\sum_{ij\in\mathcal{V}} D^3.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) = \sum_{ij\in\mathcal{E}} D^3.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) = 0\,,$$

and then, for the monomial u^2v of degree 3, the corresponding C_{11} is the same as in the constitutive definition.

Remark. The monomial u^2v^2 is not included in the space. Indeed, considering the Taylor expansion of order 4 for this monomial, we have to consider the quantity

$$C_{11} + \sum_{ij\in\mathcal{E}} D^4.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}} D^4.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})\,,$$

and check that $\sum_{ij\in\mathcal{E}} D^4.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}} D^4.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) \neq 0$ for the function $q = u^2v^2$ and the given template. Let (u_{11},v_{11}) be the coordinates of $\hat{A}_{11}$, U_{1ij} and U_{2ij} be the components of $\overrightarrow{U_{ij}}$ and V_{1ij} and V_{2ij} be the components of $\overrightarrow{V_{ij}}$, we compute the D^4 derivatives, we have

$$D^4(u_{11},v_{11}).(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) = \frac{1}{4!}\left\{\frac{\partial^4 q(u_{11},v_{11})}{\partial u^4}U_{1ij}^4 \;+\; 4\,\frac{\partial^4 q(u_{11},v_{11})}{\partial u^3\partial v}U_{1ij}^3U_{2ij}\right.$$

$$\left. +\,6\,\frac{\partial^4 q(u_{11},v_{11})}{\partial u^2\partial v^2}U_{1ij}^2,U_{2ij}^2 \;+\; 4\,\frac{\partial^4 q(u_{11},v_{11})}{\partial u\partial v^3}U_{1ij}U_{2ij}^3 \;+\; \frac{\partial^4 q(u_{11},v_{11})}{\partial v^4}U_{2ij}^4\right\}\,,$$

but this reduces to

$$D^4(u_{11},v_{11}).(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) \;=\; \frac{1}{4}\frac{\partial^4 q(u_{11},v_{11})}{\partial u^2\partial v^2}U_{1ij}^2U_{2ij}^2 \;=\; U_{1ij}^2U_{2ij}^2\,,$$

so that, after summation, since the components of the $\overrightarrow{U_{ij}}$ are respectively $(-h,-h)$, $(h,-h)$, (h,h), $(-h,h)$, we have

$$\sum_{ij\in\mathcal{V}} D^4.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) = 4h^4\,,$$

where $h = \frac{1}{2}$. Similarly we sum on set $\mathcal{E}$ where the vectors involved have as components $(0,-h),(h,0),(0,h),(-h,0)$ and we find

$$\sum_{ij\in\mathcal{E}} D^4.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) = 0\,,$$

and the sum of these two quantities is not zero, u^2v^2 is **not** in the polynomial space (if it were, this would come as a big surprise because the space would be the complete space!). Note that, since we have found u^2v and uv^2, there is no real need to check that the other monomials are not in the space because the number of monomials is equal to the dimension of the space.

As a conclusion, and following [1], the reduced element is a member of the serendipity family and the diagram of its polynomial space is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & u & & v & & \\ & u^2 & & uv & & v^2 & \\ u^2v & & & & & & uv^2 \end{array}$$

where we have the monomials of P^2 and the monomials of degree $d+1=3$ with one variable linearly involved.

As compared with the polynomial space of the complete element (the space Q^2), we miss u^2v^2. □

⋆

⋆ ⋆

Since the $p_{ij}(u,v)$s verify the same properties (e.g. Relation (30)), for all the indices ij, we have

$$4p_{ij}(\hat{A}_{11}) + \sum_{lm\in\mathcal{V}} p_{ij}(\hat{A}_{lm}) - 2\sum_{lm\in\mathcal{E}} p_{ij}(\hat{A}_{lm}) = 0\,.$$

Now, we use Relation (16) to replace the p_{ij}s by their counterpart in terms of the complete shape functions. For symmetry reasons, these relations reduce to

$$p_{ij}(u,v) = p^c_{ij}(u,v) + \alpha\, p^c_{11}(u,v) \quad \text{for} \quad ij\in\mathcal{V}\,,$$

$$\text{and} \quad p_{ij}(u,v) = p^c_{ij}(u,v) + \beta\, p^c_{11}(u,v) \quad \text{for} \quad ij\in\mathcal{E}\,,$$

in other words, there is only two coefficients. Let us fix $ij = 00$, then, we have

$$4p_{00}(\hat{A}_{11}) + \sum_{lm\in\mathcal{V}} p_{00}(\hat{S}_{lmn}) - 2\sum_{lm\in\mathcal{E}} p_{00}(\hat{A}_{lm}) = 0\,,$$

and this resumes to $\alpha = -\frac{1}{4}$, and the same (fix $ij = 10$) implies that $\beta = \frac{1}{2}$, see the following diagram:

```
-1/4   1/2   -1/4

 1/2          1/2

-1/4   1/2   -1/4
```

With these values, we have the reduced shape functions fully defined via (16), see hereafter.

Then the reduced element, seen as a patch reads

$$M(u,v) = \sum_{ij} p_{ij}(u,v) A_{ij} \,,$$

where ij lives in $\mathcal{V}$ and $\mathcal{E}$, the set of vertices and edge nodes of the reduced element, i.e., **8 indices**. We replace again the p_{ij}s by means of the p^c_{ij}s, then we have

$$M(u,v) = \sum_{ij} (\, p^c_{ij}(u,v) + \alpha_{ij} p^c_{11}(u,v)\,)\, A_{ij} \,,$$

with $\alpha_{ij} = \alpha$ or β, then this reads also

$$M(u,v) = \sum_{ij} p^c_{ij}(u,v) A_{ij} + \sum_{ij} \alpha_{ij} A_{ij} p^c_{11}(u,v) \,,$$

therefore let

$$A_{11} = \sum_{ij} \alpha_{ij} A_{ij} \,,$$

with again 8 indices so that

$$M(u,v) = \sum_{ij} p^c_{ij}(u,v) A_{ij} \,,$$

but now with **9 indices**. In other words, we have *invented* the node A_{11} with which we can define a complete element fully equivalent to the reduced element. As already seen, a complete element is equivalent to the Bézier patch

$$\sum_{i=0,2} \sum_{j=0,2} B^2_i(u) B^2_j(v) P_{ij} \,,$$

from which we obtain P_{11} and it turns out that P_{11} simply read as

$$P_{11} = \sum_{ij} \alpha_{ij} P_{ij} \,,$$

e.g., the same expression as A_{11} with 8 indices (the proof is similar to that of the 9-node triangle).

In practice. The 8-node quad is easy to analyse. Given its nodes, we find its edge control points by the formulae, here for edge $A_{00}A_{20}$,

$$P_{10} = \frac{-A_{00} + 4A_{11} - A_{20}}{2} \,.$$

Then we compute P_{11} using the above formula. Then, just use Relation (15) to have the control coefficients of the jacobian.

The shape functions. With the value of α and β and using Relation (16), it is easy to obtain the explicit expression of the shape functions. Indeed, only two of them must be made explicit since, for symmetry reasons, the other functions are evident to obtain. Therefore, we use Relation (10) to express $p^c_{00}(u,v)$, $p^c_{10}(u,v)$ and $p^c_{11}(u,v)$ and then compute $p_{00}(u,v)$ and $p_{10}(u,v)$, the two *type* functions, we find

$$p_{00}(u,v) = (1-u)(1-v)(1-2u-2v) \quad \text{and} \quad p_{10}(u,v) = 4u(1-u)(1-v)\,.$$

The two type functions are:

1	$p_{00}(u,v) = (1-u)(1-v)(1-2u-2v)$
5	$p_{10}(u,v) = 4u(1-u)(1-v)$

Type shape functions of the serendipity 8-node quad of degree 2.

By symmetry ($u \to 1-u, (u,v) \to (v,u), etc.$), the full list is the following:

1	$p_{00}(u,v) = (1-u)(1-v)(1-2u-2v)$
2	$p_{20}(u,v) = u(1-v)(-1+2u-2v)$
3	$p_{22}(u,v) = uv(-3+2u+2v)$
4	$p_{02}(u,v) = (1-u)v(-1-2u+2v)$
5	$p_{10}(u,v) = 4u(1-u)(1-v)$
6	$p_{21}(u,v) = 4uv(1-v)$
7	$p_{22}(u,v) = 4u(1-u)v$
8	$p_{01}(u,v) = 4(1-u)v(1-v)$

Shape functions of the serendipity 8-node quad of degree 2.

This concludes the case of the 8-node quadrilateral (note that this element can be defined by means of a transfinite interpolation, [23]).

The 12-node quad of degree 3. Let us turn to $d=3$. Here is the 16-node quad with four internal nodes and the numbering of the nodes is as follow:

```
03  13  23  33        03  13  23  33
02  12  22  32  ==>   02          32
01  11  21  31        01          31
00  10  20  30        00  10  20  30
```

To define the reduced element, we impose space $P^d = P^3$ to be included in the reduced polynomial space. Then, the Taylor expansion gives the serendipity relations about A_{11}, A_{21}, A_{12} and A_{22}:

$$4q(\hat{A}_{11}) + \sum_{ij \in \mathcal{V}_{11}} q(\hat{A}_{ij}) - 2 \sum_{ij \in \mathcal{E}_{11}} q(\hat{A}_{ij}) = 0\,,$$

$$4q(\hat{A}_{21}) + \sum_{ij \in \mathcal{V}_{21}} q(\hat{A}_{ij}) - 2 \sum_{ij \in \mathcal{E}_{21}} q(\hat{A}_{ij}) = 0\,,$$

$$4q(\hat{A}_{12}) + \sum_{ij \in \mathcal{V}_{12}} q(\hat{A}_{ij}) - 2 \sum_{ij \in \mathcal{E}_{12}} q(\hat{A}_{ij}) = 0\,,$$

$$4q(\hat{A}_{22}) + \sum_{ij\in\mathcal{V}_{22}} q(\hat{A}_{ij}) - 2\sum_{ij\in\mathcal{E}_{22}} q(\hat{A}_{ij}) = 0\,. \tag{33}$$

where $\hat{A}_{11}$ is the point, in the reference element $(\hat{K})$, of coordinates $(\frac{1}{3},\frac{1}{3})$, $\mathcal{V}_{11}$ is the set of the vertices in $\hat{K}$ for the sub-template centered at A_{11} and $\mathcal{E}_{11}$ is the set of the edge nodes in $\hat{K}$ for the same sub-template and, similarly, for $\hat{A}_{21}, \hat{A}_{12}$and $\hat{A}_{22}$, see the diagram.

.	.	.	.	.	.	.	.	03	13	23	.	.	13	23	33
02	12	22	.	.	12	22	32	02	12	22	.	.	12	22	32
01	11	21	.	.	11	21	31	01	11	21	.	.	11	21	31
00	10	20	.	.	10	20	30	.	.	.	.	.	.	.	.

As in the previous element, the above Serendipity relations also insure that the monomials u^3v and uv^3 are part of the reduced polynomial space.

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Proof: We return to the method used for the previous element. We consider the four sub-templates depicted in the above diagram, each of them being centered at one "central node". Given a sub-template, for example the one centered at $\hat{A}_{11}$, we consider $\mathcal{V}_{11}$, e.g., $\hat{A}_{00}, ..., \hat{A}_{02}$ and $\mathcal{E}_{11}$, e.g., $\hat{A}_{10}, ..., \hat{A}_{01}$ and we use a Taylor expansion to write the values of a function q in terms of the nodes of this two sets and the derivatives $D^1.(.), D^2.(.,.), D^3.(.,.,.)$ evaluated at $\hat{A}_{11}$. Since the expansion terminates at order 3, for a vertex, here A_{00}, we have

$$q(\hat{A}_{00}) = q(\hat{A}_{11}) + D^1.(\overrightarrow{U_{00}}) + D^2.(\overrightarrow{U_{00}},\overrightarrow{U_{00}}) + D^3.(\overrightarrow{U_{00}},\overrightarrow{U_{00}},\overrightarrow{U_{00}})\,,$$

where $\overrightarrow{U_{00}} = \overrightarrow{\hat{A}_{11}\hat{A}_{00}}$, the derivatives are evaluated at $\hat{A}_{11}$ and the symbols $D^2., D^3.$ include (for short) the coefficient, here $\frac{1}{2}$ and $\frac{1}{6}$. The same holds for the other vertices in $\mathcal{V}_{11}$, so, after summation, we have

$$\sum_{ij\in\mathcal{V}_{11}} q(\hat{A}_{ij}) = 4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{V}_{11}} D^1.(\overrightarrow{U_{ij}}) + \sum_{ij\in\mathcal{V}_{11}} D^2.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) + \sum_{ij\in\mathcal{V}_{11}} D^3.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})\,. \tag{34}$$

A similar relation holds for the nodes in $\mathcal{E}_{11}$ and the summation gives

$$\sum_{ij\in\mathcal{E}_{11}} q(\hat{A}_{ij}) = 4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{E}_{11}} D^1.(\overrightarrow{V_{ij}}) + \sum_{ij\in\mathcal{E}_{11}} D^2.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) + \sum_{ij\in\mathcal{E}_{11}} D^3.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}})\,. \tag{35}$$

where $\overrightarrow{V_{ij}} = \overrightarrow{\hat{A}_{11}\hat{A}_{ij}}$ with $ij \in \mathcal{E}_{11}$.

The derivative D^1 is a linear operator, so $\sum_{ij\in\mathcal{V}_{11}} D^1.(\overrightarrow{U_{ij}}) = D^1.(\sum_{ij\in\mathcal{V}_{11}} \overrightarrow{U_{ij}}) = 0$, similarly we have $\sum_{ij\in\mathcal{E}_{11}} D^1.(\overrightarrow{V_{ij}}) = 0$ and the same holds for the derivative D^3, and, in general, for all the odd derivatives. Now, $\overrightarrow{V_{ij}}$ can be written in terms of the $\overrightarrow{U_{ij}}$. Indeed, let us consider $\hat{A}_{10}$, we have $\hat{A}_{10} = \frac{\hat{A}_{00}+\hat{A}_{20}}{2}$ so $\overrightarrow{V_{10}} = \frac{\overrightarrow{U_{00}}+\overrightarrow{U_{20}}}{2}$ holds and we return, for this sub-template, to the result obtained in the previous element and then we obtain the above initial relation thus completing the proof.

As we did in the previous element and for a further usage, we introduce the value C_{11} defined by

$$C_{11} = \sum_{ij\in\mathcal{E}} D^2.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}} D^2.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})\,,$$

but indeed, this reads

$$C_{11} = \sum_{ij\in\mathcal{E}_{11}} D^2.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}_{11}} D^2.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) + \sum_{ij\in\mathcal{E}_{11}} D^3.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}_{11}} D^3.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})\,,$$

and the condition $C_{11} = 0$ together with $C_{21} = C_{12} = C_{22} = 0$ for the three other sub-templates are the *constitutive* definitions of the reduced element from which results the four Relations (33).

We return to C_{11}, C_{21}, C_{12} and C_{22} but, since u^3v is of degree 4, we need to examine values like (here for C_{11})

$$C_{11} = \sum_{ij\in\mathcal{E}_{11}} D^2.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}_{11}} D^2.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) + \sum_{ij\in\mathcal{E}_{11}} D^3.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}_{11}} D^3.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})$$

$$+ \sum_{ij\in\mathcal{E}_{11}} D^4.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}_{11}} D^4.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})$$

which resumes to

$$C_{11} = \sum_{ij\in\mathcal{E}_{11}} D^2.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}_{11}} D^2.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) + \sum_{ij\in\mathcal{E}_{11}} D^4.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}_{11}} D^4.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})\,,$$

and C_{11} will be zero if $\sum_{ij\in\mathcal{E}_{11}} D^4.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) - \frac{1}{2}\sum_{ij\in\mathcal{V}_{11}} D^4.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})$ vanishes. Let (u_{11}, v_{11}) be the coordinates of $\hat{A}_{11}$, U_{1ij} and U_{2ij} be the components of $\overrightarrow{U_{ij}}$ and V_{1ij} and V_{2ij} be the components of $\overrightarrow{V_{ij}}$, we compute the D^4 derivatives, we have

$$D^4(u_{11}, v_{11}).(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) = \frac{1}{4!}\left\{\frac{\partial^4 q(u_{11},v_{11})}{\partial u^4}U_{1ij}^4 \;+\; 4\,\frac{\partial^4 q(u_{11},v_{11})}{\partial u^3\partial v}U_{1ij}^3 U_{2ij}\right.$$

$$\left. +6\,\frac{\partial^4 q(u_{11},v_{11})}{\partial u^2\partial v^2}U_{1ij}^2, U_{2ij}^2 \;+\; 4\,\frac{\partial^4 q(u_{11},v_{11})}{\partial u\partial v^3}U_{1ij}U_{2ij}^3 \;+\; \frac{\partial^4 q(u_{11},v_{11})}{\partial v^4}U_{2ij}^4\right\}\,,$$

but, while $q = u^3v$, this reduces to

$$D^4(u_{11}, v_{11}).(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) \;=\; \frac{1}{4!}4\frac{\partial^4 q(u_{11},v_{11})}{\partial u^3\partial v}U_{1ij}^3 U_{2ij} \;=\; \frac{1}{6}U_{1ij}^3 U_{2ij}\,,$$

so that, after summation, since the components of the $\overrightarrow{U_{ij}}$ are respectively $(-h,-h), (h,-h), (h,h), (-h,h)$, we have

$$-\frac{1}{2}\sum_{ij\in\mathcal{V}_{11}} D^4.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) = -\frac{1}{12}(h^4 - h^4 + h^4 - h^4) \;=\; 0\,,$$

where $h = \frac{1}{2}$. Similarly we sum on set $\mathcal{E}$ where the vectors involved have as components $(0,-h), (h,0), (0,h), (-h,0)$ and we find

$$\sum_{ij\in\mathcal{E}_{11}} D^4.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) = \frac{1}{6}(0+0+0+0) = 0\,,$$

and the sum of these two quantities is zero and the same holds for the three other sums (for C_{21}, c_{12} and C_{22}) and u^3v is in the space, so it is for uv^3.

Now, with these two monomials, the dimension of the space is 12, so there is no more monomials in this space.

As a conclusion, and following [1], the reduced element is a member of the serendipity family and the diagram of its polynomial space is

$$1$$

$$u \qquad v$$

$$u^2 \qquad uv \qquad v^2$$

$$u^3 \quad u^2v \qquad uv^2 \quad v^3$$

$$u^3v \qquad\qquad\qquad\qquad uv^3$$

where we have the monomials of P^3 and the monomials of degree $d+1=4$ with one variable linearly involved. As compared with the polynomial space of the complete element (the space Q^3), we miss u^2v^2, u^3v^2, u^2v^3 and u^3v^3. □

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From the four Serendipity relations, we write a system where the unknowns are the values of the function q at the four "central" nodes, we obtain the following system:

$$\begin{bmatrix} 4 & -2 & -2 & 1 \\ -2 & 4 & 1 & -2 \\ -2 & 1 & 4 & -2 \\ 1 & -2 & -2 & 4 \end{bmatrix} \begin{pmatrix} q(\hat{A}_{11}) \\ q(\hat{A}_{21}) \\ q(\hat{A}_{12}) \\ q(\hat{A}_{22}) \end{pmatrix} = \begin{pmatrix} -q(\hat{A}_{00}) - q(\hat{A}_{20}) - q(\hat{A}_{02}) + 2q(\hat{A}_{10}) + 2q(\hat{A}_{01}) \\ -q(\hat{A}_{10}) - q(\hat{A}_{30}) - q(\hat{A}_{32}) + 2q(\hat{A}_{20}) + 2q(\hat{A}_{31}) \\ -q(\hat{A}_{01}) - q(\hat{A}_{23}) - q(\hat{A}_{03}) + 2q(\hat{A}_{13}) + 2q(\hat{A}_{02}) \\ -q(\hat{A}_{31}) - q(\hat{A}_{33}) - q(\hat{A}_{13}) + 2q(\hat{A}_{32}) + 2q(\hat{A}_{23}) \end{pmatrix}.$$

The solution (Gauss elimination by hand or `Maple`), here for $\hat{A}_{22}$, reads

$$q(\hat{A}_{22}) = \frac{1}{9}(-q(\hat{A}_{00}) - 2q(\hat{A}_{30}) - 4q(\hat{A}_{33}) - 2q(\hat{A}_{03}) + 3q(\hat{A}_{20}) + 6q(\hat{A}_{32}) + 6q(\hat{A}_{23}) + 3q(\hat{A}_{02}))$$

which corresponds to the following diagram

```
-2/9      .     2/3    -4/9

 1/3      .    [2.2]    2/3

  .       .      .       .

-1/9      .     1/3    -2/9
```

and similar solutions for the other "central" values. *In extenso*, we have

$$9q(\hat{A}_{22}) + q(\hat{A}_{00}) + 2q(\hat{A}_{30}) + 4q(\hat{A}_{33}) + 2q(\hat{A}_{03}) - 3q(\hat{A}_{20}) - 6q(\hat{A}_{32}) - 6q(\hat{A}_{23}) - 3q(\hat{A}_{02}) = 0\,,$$

$$9q(\hat{A}_{11}) + 4q(\hat{A}_{00}) + 2q(\hat{A}_{30}) + q(\hat{A}_{33}) + 2q(\hat{A}_{03}) - 6q(\hat{A}_{10}) - 3q(\hat{A}_{31}) - 3q(\hat{A}_{13}) - 6q(\hat{A}_{01}) = 0\,,$$

$$9q(\hat{A}_{21}) + 2q(\hat{A}_{00}) + 4q(\hat{A}_{30}) + 2q(\hat{A}_{33}) + q(\hat{A}_{03}) - 6q(\hat{A}_{20}) - 6q(\hat{A}_{31}) - 3q(\hat{A}_{23}) - 3q(\hat{A}_{01}) = 0\,,$$

$$9q(\hat{A}_{12}) + 2q(\hat{A}_{00}) + q(\hat{A}_{30}) + 2q(\hat{A}_{33}) + 4q(\hat{A}_{03}) - 3q(\hat{A}_{10}) - 3q(\hat{A}_{32}) - 6q(\hat{A}_{13}) - 6q(\hat{A}_{02}) = 0\,. \quad (36)$$

With these values, we have the reduced shape functions fully defined via (16), see hereafter.

Then the reduced element, seen as a patch reads

$$M(u,v) = \sum_{ij} p_{ij}(u,v) A_{ij} \,,$$

where ij lives in $\mathcal{V}$ and $\mathcal{E}$, the set of vertices and edge nodes of the reduced element, i.e., **12 indices**. We replace again the p_{ij}s by means of the p^c_{ij}s, then, with evident notations, we have

$$M(u,v) = \sum_{ij} (\, p^c_{ij}(u,v) + \alpha^{11}_{ij} p^c_{11}(u,v) + \alpha^{21}_{ij} p^c_{21}(u,v) + \alpha^{12}_{ij} p^c_{12}(u,v) + \alpha^{22}_{ij} p^c_{22}(u,v) \,) \, A_{ij} \,,$$

with, for example, α^{22}_{ij}, the coefficients in the above diagram and similar values for the 3 other nodes. Then, this reads also

$$M(u,v) = \sum_{ij} p^c_{ij}(u,v) A_{ij} + \sum_{ij} \alpha^{11}_{ij} A_{ij} p^c_{11}(u,v) + \sum_{ij} \alpha^{21}_{ij} A_{ij} p^c_{21}(u,v)$$

$$+ \sum_{ij} \alpha^{12}_{ij} A_{ij} p^c_{12}(u,v) + \sum_{ij} \alpha^{22}_{ij} A_{ij} p^c_{22}(u,v) \,,$$

therefore let

$$A_{11} = \sum_{ij} \alpha^{11}_{ij} A_{ij}, \quad A_{21} = \sum_{ij} \alpha^{21}_{ij} A_{ij}, \ldots,$$

again with **12 indices**, so that the patch is

$$M(u,v) = \sum_{ij} p^c_{ij}(u,v) A_{ij} \,,$$

but now with **16 indices**. In other words, we have *invented* the nodes A_{11}, A_{21}, A_{12} and A_{22} with which we can define a complete element fully equivalent to the reduced element. As already seen, a complete element is equivalent to the Bézier patch

$$\sum_{i=0,3} \sum_{j=0,3} B^3_i(u) B^3_j(v) P_{ij} \,,$$

from which we obtain P_{11}, P_{21}, P_{12} and P_{22} and it turns out that P_{11} simply read as

$$P_{11} = \sum_{ij} \alpha^{11}_{ij} P_{ij} \,,$$

with, here, **12 indices**, e.g., the same expression as A_{11} and similar values for the 3 other control points, see [20].

In practice. The 12-node quad is easy to analyse. Given its nodes, we find its edge control points by the formulae, here for edge $A_{00}A_{30}$,

$$P_{10} = \frac{-5A_{00} + 18A_{10} - 9A_{20} + 2A_{30}}{6} \quad \text{and} \quad P_{20} = \frac{2A_{00} - 9A_{10} + 18A_{20} - 5A_{30}}{6} \,.$$

Then we compute P_{11}, P_{21}, P_{12} and P_{22} using the above formulae. Then, just use Relation (15) to have the control coefficients of the jacobian.

The shape functions. With the values of α_{ij}^{kl} and using Relation (16), it is easy to obtain the explicit expression of the shape functions. Indeed, only two of them must be made explicit since, for symmetry reasons, the other functions are evident to obtain. Therefore, we use Relation (10) to express $p_{00}^c(u,v)$, $p_{10}^c(u,v)$ and $p_{11}^c(u,v), p_{21}^c(u,v)$, $p_{12}^c(u,v)$ and $p_{22}^c(u,v)$ and then compute $p_{00}(u,v)$ and $p_{10}(u,v)$, the two *type* functions, we find

$$p_{00}(u,v) = \frac{9}{2}(1-u)(1-v)(\frac{2}{9} - u - v + u^2 + v^2) \quad \text{and} \quad p_{10}(u,v) = \frac{9}{2}u(1-u)(2-3u)(1-v)\,.$$

The two type functions are:

1	$p_{00}(u,v) = \frac{9}{2}(1-u)(1-v)(\frac{2}{9} - u - v + u^2 + v^2)$
5	$p_{10}(u,v) = \frac{9}{2}u(1-u)(2-3u)(1-v)$

Type shape functions of the serendipity 12-node quad of degree 3.

By symmetry ($u \to 1-u, (u,v) \to (v,u), etc.$), the full list is the following:

1	$p_{00}(u,v) = \frac{9}{2}(1-u)(1-v)(\frac{2}{9} - u - v + u^2 + v^2)$
2	$p_{30}(u,v) = \frac{9}{2}u(1-v)(\frac{2}{9} - u - v + u^2 + v^2)$
3	$p_{33}(u,v) = \frac{9}{2}uv(\frac{2}{9} - u - v + u^2 + v^2)$
4	$p_{03}(u,v) = \frac{9}{2}(1-u)v(\frac{2}{9} - u - v + u^2 + v^2)$
5	$p_{10}(u,v) = \frac{9}{2}u(1-u)(2-3u)(1-v)$
6	$p_{20}(u,v) = \frac{9}{2}u(1-u)(3u-1)(1-v)$
7	$p_{31}(u,v) = \frac{9}{2}v(1-v)(2-3v)u$
8	$p_{32}(u,v) = \frac{9}{2}v(1-v)(3v-1)u$
9	$p_{23}(u,v) = \frac{9}{2}u(1-u)(3u-1)v$
10	$p_{13}(u,v) = \frac{9}{2}u(1-u)(2-3u)v$
11	$p_{02}(u,v) = \frac{9}{2}v(1-v)(3v-1)(1-u)$
12	$p_{01}(u,v) = \frac{9}{2}v(1-v)(2-3v)(1-u)$

Shape functions of the serendipity 12-node quad of degree 3.

This concludes the case of the 12-node quad (note that this element can also be defined by means of a transfinite interpolation, [23]).

The 17-node quad of degree 4. Let us turn to $d = 4$. The 25-node quad includes 9 internal nodes structured as follow:

```
04   14   24   34   44         04   14   24   34   44
03   13   23   33   43         03                  43
02   12   22   32   42   ==>   02        22        42
01   11   21   31   41         01                  41
00   10   20   30   40         00   10   20   30   40
```

We follow the same method using a Taylor expansion and looking for $P^d = P^4$ on the sub-templates (see the diagram) centered at the "central" nodes. As for the sub-template centered at $\hat{A}_{11}$, we write the vertex $\hat{A}_{00}$ and the other vertices and, after summation (the terms in D^1 and in D^3 sum to zero), we have

$$q(\hat{A}_{00}) + q(\hat{A}_{20}) + q(\hat{A}_{22}) + q(\hat{A}_{02}) = 4q(\hat{A}_{11})$$

$$+D^2.(\overrightarrow{U_{00}},\overrightarrow{U_{00}}) + D^2.(\overrightarrow{U_{20}},\overrightarrow{U_{20}}) + D^2.(\overrightarrow{U_{22}},\overrightarrow{U_{22}}) + D^2.(\overrightarrow{U_{02}},\overrightarrow{U_{02}})$$

$$+D^4.(\overrightarrow{U_{00}},\overrightarrow{U_{00}},\overrightarrow{U_{00}},\overrightarrow{U_{00}})+D^4.(\overrightarrow{U_{20}},\overrightarrow{U_{20}},\overrightarrow{U_{20}},\overrightarrow{U_{20}})+D^4.(\overrightarrow{U_{22}},\overrightarrow{U_{22}},\overrightarrow{U_{22}},\overrightarrow{U_{22}})+D^4.(\overrightarrow{U_{02}},\overrightarrow{U_{02}},\overrightarrow{U_{02}},\overrightarrow{U_{02}})\,. \tag{37}$$

written as

$$\sum_{ij\in\mathcal{V}_{11}} q(\hat{A}_{ij}) = 4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{V}_{11}} D^2.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) + \sum_{ij\in\mathcal{V}_{11}} D^4.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})\,,$$

with the same notations as before.

For the nodes in $\mathcal{E}_{11}$, we have a similar relation

$$\sum_{ij\in\mathcal{E}_{11}} q(\hat{A}_{ij}) = 4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{E}_{11}} D^2.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) + \sum_{ij\in\mathcal{E}_{11}} D^4.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}})\,.$$

As for the previous element, we have

$$\sum_{ij\in\mathcal{E}_{11}} D^2.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) = \frac{1}{2}\sum_{ij\in\mathcal{V}_{11}} D^2.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})\,,$$

from these we combine the two above lines so as to cancel the terms in D^2 and we obtain

$$\sum_{ij\in\mathcal{V}_{11}} q(\hat{A}_{ij})-2\sum_{ij\in\mathcal{E}_{11}} q(\hat{A}_{ij}) = -4q(\hat{A}_{11})+\sum_{ij\in\mathcal{V}_{11}} D^4.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}})-2\sum_{ij\in\mathcal{E}_{11}} D^4.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}})\,,$$

But the D^4 derivatives are a constant, let us define $C_{11} = \sum_{ij\in\mathcal{V}_{11}} D^4.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) - 2\sum_{ij\in\mathcal{E}_{11}} D^4.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}})$, then we have the relation

$$C_{11} = 4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{V}_{11}} q(\hat{A}_{ij}) - 2\sum_{ij\in\mathcal{E}_{11}} q(\hat{A}_{ij})\,. \tag{38}$$

Then, we have the relation about $\hat{A}_{11}$ for the first sub-template

```
.    .    .    .    .
.    .    .    .    .
02   12   22   .    .
01   11   21   .    .
00   10   20   .    .
```

and we repeat the same process for all the other sub-templates. As a consequence, we obtain 9 relations, say

$$C_{11} = 4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{V}_{11}} q(\hat{A}_{ij}) - 2\sum_{ij\in\mathcal{E}_{11}} q(\hat{A}_{ij})\,,$$

$$C_{21} = 4q(\hat{A}_{21}) + \sum_{ij\in\mathcal{V}_{21}} q(\hat{A}_{ij}) - 2\sum_{ij\in\mathcal{E}_{21}} q(\hat{A}_{ij})\,,$$

$$\ldots$$

$$C_{22} = 4q(\hat{A}_{22}) + \sum_{ij\in\mathcal{V}_{22}} q(\hat{A}_{ij}) - 2\sum_{ij\in\mathcal{E}_{22}} q(\hat{A}_{ij})\,.$$

Now since we have $C_{11} = C_{21} = ... = C_{22}$, therefore only 8 equations, we consider the "central" function (index $_{22}$) as known so as to have only 8 unknowns and a 8×8 system in hand (and the reduced element has 17 nodes, one being internal).

The above system is the constitutive Serendipity relation. We will show that the monomials u^4v and uv^4 are also covered. For monomial u^4v, the Taylor expansion of degree 5 gives:

$$C_{11} = \sum_{ij\in\mathcal{V}_{11}} D^4.(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}) - 2 \sum_{ij\in\mathcal{E}_{11}} D^4.(\overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}) \,,$$

as

$$\sum_{ij\in\mathcal{V}_{11}} D^5.(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}) = \sum_{ij\in\mathcal{E}_{11}} D^5.(\overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}) = 0 \,.$$

Now

$$D^4(u_{11}, v_{11}).(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}) = \frac{1}{4!}\left\{ \frac{\partial^4 q(u_{11},v_{11})}{\partial u^4} U_{1ij}^4 \,+\, 4\,\frac{\partial^4 q(u_{11},v_{11})}{\partial u^3 \partial v} U_{1ij}^3 U_{2ij} \right.$$

$$\left. +\,6\,\frac{\partial^4 q(u_{11},v_{11})}{\partial u^2 \partial v^2} U_{1ij}^2, U_{2ij}^2 \,+\, 4\,\frac{\partial^4 q(u_{11},v_{11})}{\partial u \partial v^3} U_{1ij} U_{2ij}^3 \,+\, \frac{\partial^4 q(u_{11},v_{11})}{\partial v^4} U_{2ij}^4 \right\} \,,$$

but each partial derivatives are linear, more precisely:

$$\frac{\partial^4 q(u_{11},v_{11})}{\partial u^4} = 4!\, v_{11} \,,$$

$$\frac{\partial^4 q(u_{11},v_{11})}{\partial u^3 \partial v} = 3!\, u_{11} \,,$$

and the other partials are null. Therefore

$$D^4(u_{11}, v_{11}).(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}) = v_{11} U_{1ij}^4 \,+\, u_{11} U_{1ij}^3 U_{2ij} \,.$$

Hence,

$$\sum_{ij\in\mathcal{V}_{11}} D^4.(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}) = \sum_{ij\in\mathcal{V}_{11}} \left\{ v_{11} U_{1ij}^4 \,+\, u_{11} U_{1ij}^3 U_{2ij} \right\} = \sum_{ij\in\mathcal{V}_{11}} \left\{ v_{11} U_{1ij}^4 \right\} = 4 v_{11} h^4 \,,$$

now, we consider

$$\sum_{ij\in\mathcal{E}_{11}} D^4.(\overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}) = \sum_{ij\in\mathcal{V}_{11}} \left\{ v_{11} V_{1ij}^4 \,+\, u_{11} V_{1ij}^3 V_{2ij} \right\} = \sum_{ij\in\mathcal{V}_{11}} \left\{ v_{11} V_{1ij}^4 \right\} = 2 v_{11} h^4 \,.$$

Then, we have $C_{11} = 0$ which is a particular case of the constitutive relation.

As a conclusion, and following [1], the reduced element is a member of the serendipity family and the diagram of its polynomial space is

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & u & & v & & & & \\
 & & & u^2 & & uv & & v^2 & & & \\
 & & u^3 & & u^2v & & uv^2 & & v^3 & & \\
 & u^4 & & u^3v & & u^2v^2 & & uv^3 & & v^4 & \\
u^4v & & & & & & & & & & uv^4
\end{array}$$

where we have the monomials of P^4 and the monomials of degree $d+1=5$ with one variable linearly involved. As compared with the polynomial space of the complete element (the space Q^4), we miss the 8 monomials among the u^iv^js with $i+j \geq 5$ apart u^4v and uv^4.

Now, we solve the Serendipity system to finding the Serendipity relationships. The first equation results from $C_{11}=C_{21}$, and we find

$$6q(\hat{A}_{11}) - 6q(\hat{A}_{21}) + 2q(\hat{A}_{31}) - 3q(\hat{A}_{12}) + 3q(\hat{A}_{22}) - q(\hat{A}_{32}) =$$

$$-q(\hat{A}_{00}) + 3q(\hat{A}_{10}) - 3q(\hat{A}_{20}) + q(\hat{A}_{30}) + 2q(\hat{A}_{01}) - q(\hat{A}_{02}) .$$

Once the 7 other equations, see [20], are obtained, we solve the system, using `Maple`, and the solution, by symmetry, can be deduced from only two precise (*type*) solutions, the one in $\hat{A}_{11}$ and the other in $\hat{A}_{21}$, we have

$$q(\hat{A}_{11}) = \frac{1}{64}\Big\{-27q(\hat{A}_{00}) + 48q(\hat{A}_{10}) + 48q(\hat{A}_{01}) - 3q(\hat{A}_{40}) + 16q(\hat{A}_{41}) + 16q(\hat{A}_{14}) + 5q(\hat{A}_{44})$$

$$-3q(\hat{A}_{04}) - 18q(\hat{A}_{20}) - 18q(\hat{A}_{02}) + 36q(\hat{A}_{22}) - 18q(\hat{A}_{42}) - 18q(\hat{A}_{24})\Big\} , \tag{39}$$

and

$$q(\hat{A}_{21}) = \frac{1}{16}\Big\{-3q(\hat{A}_{00}) + 6q(\hat{A}_{20}) + 8q(\hat{A}_{01}) - 6q(\hat{A}_{02}) + 12q(\hat{A}_{22}) - 3q(\hat{A}_{40}) + 8q(\hat{A}_{41}) - 6q(\hat{A}_{42})$$

$$-2q(\hat{A}_{24}) + 1q(\hat{A}_{44}) + 1q(\hat{A}_{04})\Big\} , \tag{40}$$

and the corresponding diagrams are

-3/64	16/64	-18/64	0	5/64		1/16	0	-2/16	0	1/16
0				0		0			0	
-18/64		36/64		-18/64		-6/16		12/16		-6/16
48/64	[1.1]			16/64		8/16		[2.1]		8/16
-27/64	48/64	-18/64	0	-3/64		-3/16	0	6/16	0	-3/16

Using these coefficients, we have the reduced shape functions fully defined via (16), see hereafter.

Then the reduced element, seen as a patch reads

$$M(u,v) = \sum_{ij} p_{ij}(u,v)A_{ij} ,$$

where ij denotes the indices of the (4) vertices, the (12) edge nodes and the (1) internal node of the reduced element, i.e., **17 indices**. We replace again the p_{ij}s by means of the p^c_{ij}s, then, with evident notations, we have

$$M(u,v) = \sum_{ij}\left\{ p^c_{ij}(u,v) + \sum_{kl} \alpha^{kl}_{ij} p^c_{kl}(u,v) \right\} A_{ij} ,$$

with, for example, α_{ij}^{11} and α_{ij}^{21}, the coefficients in the above diagrams. Then, this also reads

$$M(u,v) = \sum_{ij} p_{ij}^c(u,v)A_{ij} + \sum_{ij}\sum_{kl} \alpha_{ij}^{kl} A_{ij} p_{kl}^c(u,v) \; = \; \sum_{ij} p_{ij}^c(u,v)A_{ij} + \sum_{kl}\sum_{ij} \alpha_{ij}^{kl} A_{ij} p_{kl}^c(u,v)\,,$$

therefore let

$$A_{kl} = \sum_{ij} \alpha_{ij}^{kl} A_{ij}\,,$$

again with **17 indices**, so that the patch is

$$M(u,v) = \sum_{ij} p_{ij}^c(u,v)A_{ij}\,,$$

but now with **25 indices**. In other words, we have *invented* the nodes $A_{11}, A_{21}, ...$ with which we can define a complete element fully equivalent to the reduced element. As already seen, a complete element is equivalent to the Bézier patch

$$\sum_{i=0,4}\sum_{j=0,4} B_i^4(u)B_j^4(v)P_{ij}\,,$$

from which we obtain $P_{11}, P_{21}, ...$ and the relations are not those of the above nodes.

In practice. The 17-node quad is easy to analyse. Given its nodes, we find its edge control points by the formulae, here for edge $A_{00}A_{40}$,

$$P_{10} = \frac{-13A_{00} + 48A_{10} - 36A_{20} + 16A_{30} - 3A_{40}}{12}\,,$$

$$P_{20} = \frac{13A_{00} - 64A_{10} + 120A_{20} - 64A_{30} + 13A_{40}}{18}\,,$$

$$P_{30} = \frac{-3A_{00} + 16A_{10} - 36A_{20} + 48A_{30} - 13A_{40}}{12}\,.$$

Then we compute $P_{11}, P_{21}, ...$ by solving the corresponding system. Then, just use Relation (15) to have the control coefficients of the jacobian.

The shape functions. With the values of α_{ij}^{kl} and using Relation (16), it is easy to obtain the explicit expression of the shape functions. Indeed, only four of them must be made explicit since, for symmetry reasons, the other functions are evident to obtain. Therefore, we use Relation (10) to express $p_{00}^c(u,v)$, $p_{10}^c(u,v)$, $p_{20}^c(u,v)$ and $p_{22}^c(u,v)$ with $p_{11}^c(u,v)$, $p_{21}^c(u,v), ..., p_{12}^c(u,v)$ and then compute $p_{00}(u,v), p_{10}(u,v), p_{20}(u,v)$ and $p_{22}(u,v)$, the four *type* functions, we find

$$p_{00}(u,v) = 1/3(1-u)(1-v)(3 - 22u - 22v + 48u^2 + 12uv + 48v^2 - 32u^3 - 32v^3)\,,$$

$$p_{10}(u,v) = 16/3(u)(1-u)(1-2u)(3-4u)(1-v)\,,$$

$$p_{20}(u,v) = 4u(1-u)(1-v)(-3 + 16u - 2v - 16u^2)\,,$$

$$p_{22}(u,v) = 16u(1-u)v(1-v)\,.$$

The four type functions are:

1	$p_{00}(u,v) = \frac{1}{3}(1-u)(1-v)(3-22u-22v+48u^2+12uv+48v^2-32u^3-32v^3)$
5	$p_{10}(u,v) = \frac{16}{3}u(1-u)(1-2u)(3-4u)(1-v)$
6	$p_{20}(u,v) = 4u(1-u)(1-v)(-3+16u-2v-16u^2)$
17	$p_{22}(u,v) = 16u(1-u)v(1-v)$

Type shape functions of the serendipity 17-node quad of degree 4.

By symmetry $(u \to 1-u, (u,v) \to (v,u), etc.)$, the full list is the following:

1	$p_{00}(u,v) = \frac{1}{3}(1-u)(1-v)(3-22u-22v+48u^2+12uv+48v^2-32u^3-32v^3)$
2	$p_{40}(u,v) = \frac{1}{3}u(1-v)(-3+22u-10v-48u^2-12uv+48v^2+32u^3-32v^3)$
3	$p_{44}(u,v) = \frac{1}{3}uv(3+10u+10v-48u^2+12uv-48v^2+32u^3+32v^3)$
4	$p_{04}(u,v) = \frac{1}{3}(1-u)v(-3-10u+22v+48u^2-12uv-48v^2-32u^3+32v^3)$
5	$p_{10}(u,v) = \frac{16}{3}u(1-u)(1-2u)(3-4u)(1-v)$
6	$p_{20}(u,v) = 4u(1-u)(1-v)(-3+16u-2v-16u^2)$
7	$p_{30}(u,v) = \frac{16}{3}u(1-u)(1-2u)(1-4u)(1-v)$
8	$p_{41}(u,v) = \frac{16}{3}uv(1-v)(1-2v)(3-4v)$
9	$p_{42}(u,v) = 4v(1-v)u(-3+16v-2u-16v^2)$
10	$p_{43}(u,v) = \frac{16}{3}uv(1-v)(1-2v)(1-4v)$
11	$p_{34}(u,v) = \frac{16}{3}u(1-u)(1-2u)(1-4u)v$
12	$p_{24}(u,v) = 4u(1-u)v(-3+16u-2v-16u^2)$
13	$p_{14}(u,v) = \frac{16}{3}u(1-u)(1-2u)(3-4u)v$
14	$p_{03}(u,v) = \frac{16}{3}(1-u)v(1-v)(1-2v)(1-4v)$
15	$p_{02}(u,v) = 4v(1-v)(1-u)(-3+16v-2u-16v^2)$
16	$p_{01}(u,v) = \frac{16}{3}(1-u)v(1-v)(1-2v)(3-4v)$
17	$p_{22}(u,v) = 16u(1-u)v(1-v)$

Shape functions of the serendipity 17-node quad of degree 4.

This concludes the case of the 17-node quad.

The 24-node quad of degree 5. Let us turn to $d = 5$. The 36-node quad includes 16 internal nodes structured as follow:

```
05  15  25  35  45  55        05  15  25  35  45  55
04  14  24  34  44  54        04                  54
03  13  23  33  43  53        03      23  33      53
02  12  22  32  42  52  ==>   02      22  32      52
01  11  21  31  41  51        01                  51
00  10  20  30  40  50        00  10  20  30  40  50
```

A serendipity quad of degree 5 can be found in [1] and [15] but this element has 23 nodes therefore only 3 internal nodes. As a consequence, the internal nodes cannot be symmetrically located. While this seems appropriate for a bidimensional element, we are convinced that it is not suitable in three dimensions if we consider the faces of an hexahedron of degree 5. This is the reason why we like to find a different element, now, with 24 nodes.

As a matter of fact, what we did for the quad of degree 4 applies here for all the sub-templates, so that we have relations like

$$\sum_{ij\in\mathcal{V}_{11}} q(\hat{A}_{ij}) - 2\sum_{ij\in\mathcal{E}_{11}} q(\hat{A}_{ij}) = -4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{V}_{11}} D^4.(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) - 2\sum_{ij\in\mathcal{E}_{11}} D^4.(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}),$$

But now the D^4 derivatives are not constant but are a polynomial of degree 1 in terms of the coordinates of $\hat{A}_{11}$ for the given configuration of the vectors $\overrightarrow{U_{ij}}$ and $\overrightarrow{V_{ij}}$ involved in the sub-template. Let us define $C_{11} = \sum_{ij\in\mathcal{V}_{11}} D^4.(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}) - 2\sum_{ij\in\mathcal{E}_{11}} D^4.(\overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}, \overrightarrow{V_{ij}})$, so that the previous relation reads

$$C_{11} = 4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{V}_{11}} q(\hat{A}_{ij}) - 2\sum_{ij\in\mathcal{E}_{11}} q(\hat{A}_{ij})\,. \tag{41}$$

Let (u_{11}, v_{11}) be the coordinates of $\hat{A}_{11}$, U_{1ij} and U_{2ij} be the components of $\overrightarrow{U_{ij}}$ and V_{1ij} and V_{2ij} be the components of $\overrightarrow{V_{ij}}$, we compute the D^4 derivatives, we have

$$C_{11} = C_{11}(u_{11}, v_{11}) = \sum_{ij\in\mathcal{V}_{11}} D^4(u_{11}, v_{11}).(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}) - 2\sum_{ij\in\mathcal{E}_{11}} D^4(u_{11}, v_{11}).(\overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}, \overrightarrow{V_{ij}}, \overrightarrow{V_{ij}})\,,$$

and

$$D^4(u_{11}, v_{11}).(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}) = \frac{\partial^4 q(u_{11}, v_{11})}{\partial u^4} U_{1ij}^4 + 4\,\frac{\partial^4 q(u_{11}, v_{11})}{\partial u^3 \partial v} U_{1ij}^3 U_{2ij}$$
$$+ 6\,\frac{\partial^4 q(u_{11}, v_{11})}{\partial u^2 \partial v^2} U_{1ij}^2 U_{2ij}^2 + 4\,\frac{\partial^4 q(u_{11}, v_{11})}{\partial u \partial v^3} U_{1ij} U_{2ij}^3 + \frac{\partial^4 q(u_{11}, v_{11})}{\partial v^4} U_{2ij}^4\,,$$

simply written as (with C_4^k the binomial coefficients)

$$D^4(u_{11}, v_{11}).(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}) = \sum_{k=0}^{4} C_4^k \frac{\partial^4 q(u_{11}, v_{11})}{\partial u^k \partial v^{4-k}} U_{1ij}^k U_{2ij}^{4-k}\,.$$

Since the degree of q is 5, we have

$$\frac{\partial^4 q(u_{11}, v_{11})}{\partial u^k \partial v^{4-k}} = \alpha_k u_{11} + \beta_k v_{11} + \gamma_k\,,$$

with some coefficients α_k, β_k and γ_k. The idea is to consider C_{21} and, more precisely, the term $D^4(u_{21}, v_{21}).(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}})$ and to look at the difference

$$D^4(u_{21}, v_{21}).(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}) - D^4(u_{11}, v_{11}).(\overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}}, \overrightarrow{U_{ij}})\,,$$

this reads

$$\sum_k C_4^k \left\{(\alpha_k u_{21} + \beta_k v_{21} + \gamma_k) - (\alpha_k u_{11} + \beta_k v_{11} + \gamma_k)\right\} U_{1ij}^k U_{2ij}^{4-k}\,,$$

but $v_{21} = v_{11}$ and $u_{21} - u_{11} = \frac{1}{5}$, so this resumes to

$$\sum_k C_4^k \alpha_k (u_{21} - u_{11}) U_{1ij}^k U_{2ij}^{4-k} = \sum_k C_4^k \frac{\alpha_k}{5} U_{1ij}^k U_{2ij}^{4-k}\,,$$

which is then independent of the position. For the other term in C_{11} and C_{21}, we have the same result. In the other direction, say by considering $C_{12} - C_{11}$, we also have the same conclusion. Therefore, we impose the equality between the difference of two consecutive C_{ij} in both directions. This results in 8 relations in the u-directions, 2 by line where v is constant. Then we impose the equality between the four lines v constant, therefore 3 more relations and, by summation, we have 11 relations in the u-direction. We also have *a priori* 11 similar relations in the v-direction. But some are redundant. Indeed, once the relations for the lines (in u) are imposed, the 2 relations for the first column (in v) induce the 9 others. Therefore we have 13 relations and 12 unknowns. It means that there exists a relation about the lines which is still redundant and that gives a dependency of one central node with the free nodes.

Now, we show the inclusion of the monomials u^5v and uv^5 in the reduced polynomial space using the Serendipity relations (like $C_{12} - C_{11} = constante$). The Taylor expansion of degree 6 of monomial u^5v leads to

$$C_{11} = \sum_{ij\in\mathcal{V}_{11}} D^4(u_{11},v_{11}).(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) - 2\sum_{ij\in\mathcal{E}_{11}} D^4(u_{11},v_{11}).(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}})$$

$$+ \sum_{ij\in\mathcal{V}_{11}} D^6(u_{11},v_{11}).(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) - 2\sum_{ij\in\mathcal{E}_{11}} D^6(u_{11},v_{11}).(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}})\,.$$

In this expression, partials in D^6 are constant while those in D^4 are polynomials of degree 2.

$$\sum_{ij\in\mathcal{V}_{11}} D^4(u_{11},v_{11}).(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) = \sum_{ij\in\mathcal{V}_{11}} \left\{5u_{11}v_{11}U_{1ij}^4 \,+\, 10u_{11}^2U_{1ij}^3U_{2ij}\right\}$$

$$= \sum_{ij\in\mathcal{V}_{11}} \left\{5u_{11}v_{11}U_{1ij}^4\right\} = 20u_{11}v_{11}h^4\,,$$

Now

$$\sum_{ij\in\mathcal{E}_{11}} D^4(u_{11},v_{11}).(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}}) = \sum_{ij\in\mathcal{V}_{11}} \left\{5u_{11}v_{11}V_{1ij}^4 \,+\, 10u_{11}^2V_{1ij}^3V_{2ij}\right\}$$

$$= \sum_{ij\in\mathcal{V}_{11}} \left\{5u_{11}v_{11}V_{1ij}^4\right\} == 10u_{11}v_{11}h^4\,,$$

then

$$C_{11} = \sum_{ij\in\mathcal{V}_{11}} D^6(u_{11},v_{11}).(\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}},\overrightarrow{U_{ij}}) - 2\sum_{ij\in\mathcal{E}_{11}} D^6(u_{11},v_{11}).(\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}},\overrightarrow{V_{ij}})\,,$$

which depends only on U_{ij} and V_{ij}. Hence, we obtain $C_{12} - C_{11} = 0$ which is again a particular case of the Serendipity relations.

As a conclusion, and following [1], the reduced element is a member of the serendipity family and the diagram of its polynomial space is

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & u & & v & & \\
 & u^2 & & uv & & v^2 & \\
u^3 & u^2v & & & uv^2 & v^3 & \\
u^4 & u^3v & u^2v^2 & & uv^3 & v^4 & \\
u^5 & u^4v & u^3v^2 & u^2v^3 & uv^4 & v^5 & \\
u^5v & & & u^3v^3 & & & uv^5
\end{array}$$

where we have the monomials of P^5 and the monomials of degree $d+1=6$ with one variable linearly involved, however, **plus** u^3v^3. As compared with the polynomial space of the complete element (the space Q^5), we miss the 12 monomials among the u^iv^js with $i+j \geq 6$ apart u^5v, u^3v^3 and uv^5.

Remark 1: An alternative solution can be found by selecting, as internal nodes, the other possible symmetric configuration $(11, 41, 44, 14)$.

To solve the Serendipity system, we give here the main lines of the solution (see [20] for a detailed description).

First of all, we write the 11 relations related to the lines, for example, the first results from $C_{21} - C_{11} = C_{31} - C_{21}$ using relations like (41), it holds

$$(R1) \qquad 12q(\hat{A}_{21}) + 4q(\hat{A}_{10}) + 4q(\hat{A}_{30}) + 4q(\hat{A}_{32}) + 4q(\hat{A}_{12}) - 6q(\hat{A}_{20}) - 8q(\hat{A}_{31}) - 6q(\hat{A}_{22})$$
$$-8q(\hat{A}_{11}) - q(\hat{A}_{00}) - q(\hat{A}_{02}) + 2q(\hat{A}_{01}) - q(\hat{A}_{40}) - q(\hat{A}_{42}) + 2q(\hat{A}_{41}) = 0\,.$$

With the 10 other equalities, we obtain a system where the matrix is a 11×12 matrix. Then we consider the augmented matrix made up of the previous with an additional column representing terms independent of the unknowns, therefore a 11×13 matrix. The rank of this augmented matrix is 10 (as 2 relations relative to the columns are not yet considered). An LU decomposition of this matrix reveals a line of zeros apart from the entry corresponding to the augmented column. In order to have a solution, this last entry must be null from which we have a dependency between the four (central) unknowns (indices $_{22}$, $_{32}$, $_{23}$, $_{33}$), we find

$$\frac{1}{5}q(\hat{A}_{00}) - q(\hat{A}_{20}) + q(\hat{A}_{30}) - \frac{1}{5}q(\hat{A}_{50}) - \frac{1}{5}q(\hat{A}_{05}) + \frac{1}{5}q(\hat{A}_{55}) + q(\hat{A}_{25}) - q(\hat{A}_{35})$$
$$- 5q(\hat{A}_{32}) + 5q(\hat{A}_{22}) - q(\hat{A}_{02}) + q(\hat{A}_{52}) + 5q(\hat{A}_{33}) - 5q(\hat{A}_{23}) + q(\hat{A}_{03}) - q(\hat{A}_{53}) = 0\,. \qquad (42)$$

Therefore, assuming the above condition and using the 10 other equations of the above lines completed by the 2 equations in the v directions, we obtain a 12×12 matrix of rank 12. Using `Maple`, we obtain the solution. It is as follow (for short read aij like $q(\hat{A}_{ij})$:

$$a11 = 4/5a10 + 22/45a30 - 14/9a32 - 38/45a20 + 22/9a22 - 74/225a00 - 38/45a02 + 4/5a01$$
$$-26/225a50 - 2/45a52 + 1/5a51 + 10/9a33 - 14/9a23 + 22/45a03 - 2/45a53 - 2/45a35$$
$$+1/5a15 - 2/45a25 - 26/225a05 + 1/225a55$$
$$a21 = 4/15a30 - 4/3a32 + 8/3a22 - 8/75a00 - 16/15a02 + 3/5a01 - 4/25a50 - 4/15a52$$
$$+2/5a51 + 4/3a33 - 2a23 + 2/3a03 - 4/15a35 + 1/3a25 - 7/75a05 + 2/75a55$$
$$a31 = 3/5a30 - 1/3a32 - 1/3a20 + 5/3a22 - 1/25a00 - 13/15a02 + 2/5a01 - 17/75a50$$
$$-7/15a52 + 3/5a51 + a33 - 5/3a23 + 3/5a03 + 1/15a53 - 4/15a35 + 1/3a25$$
$$-7/75a05 + 2/75a55$$
$$a41 = -4/9a30 + 4/9a32 + 4/45a20 + 4/9a22 - 8/225a00 - 4/9a02 + 1/5a01 + 4/5a40$$
$$-92/225a50 - 4/9a52 + 4/5a51 + 4/9a33 - 8/9a23 + 16/45a03 + 4/45a53 - 4/9a35$$
$$+16/45a25 + 1/5a45 - 17/225a05 - 8/225a55$$
$$a42 = -4/5a30 + 4/3a32 + 2/5a20 - 2/3a22 - 1/25a00 + 1/15a02 + 3/5a40 - 4/25a50$$
$$+4/15a52 - 8/15a35 + 4/15a25 + 2/5a45 - 2/75a05 - 8/75a55$$
$$a43 = -8/15a30 + 4/15a20 - 2/75a00 + 2/5a40 - 8/75a50 + 4/3a33 - 2/3a23 + 1/15a03$$
$$+4/15a53 - 4/5a35 + 2/5a25 + 3/5a45 - 1/25a05 - 4/25a55$$

$$a44 = -4/9a30 + 4/9a32 + 16/45a20 - 8/9a22 - 17/225a00 + 16/45a02 + 1/5a40 - 8/225a50$$
$$+4/45a52 + 4/9a33 + 4/9a23 - 4/9a03 + 1/5a04 - 4/9a53 + 4/5a54 - 4/9a35 + 4/45a25$$
$$+4/5a45 - 8/225a05 - 92/225a55$$
$$a34 = -4/15a30 + a32 + 1/3a20 - 5/3a22 - 7/75a00 + 3/5a02 + 2/75a50 + 1/15a52 - 1/3a33$$
$$+5/3a23 - 13/15a03 + 2/5a04 - 7/15a53 + 3/5a54 + 3/5a35 - 1/3a25 - 1/25a05 - 17/75a55$$
$$a24 = -4/15a30 + 4/3a32 + 1/3a20 - 2a22 - 7/75a00 + 2/3a02 + 2/75a50 - 4/3a33 + 8/3a23$$
$$-16/15a03 + 3/5a04 - 4/15a53 + 2/5a54 + 4/15a35 - 8/75a05 - 4/25a55$$
$$a14 = 1/5a10 - 2/45a30 + 10/9a32 - 2/45a20 - 14/9a22 - 26/225a00 + 22/45a02 + 1/225a50$$
$$-2/45a52 - 14/9a33 + 22/9a23 - 38/45a03 + 4/5a04 - 2/45a53 + 1/5a54 + 22/45a35$$
$$+4/5a15 - 38/45a25 - 74/225a05 - 26/225a55$$
$$a13 = 2/5a10 + 4/15a30 - 8/15a20 - 8/75a00 - 2/75a50 - 2/3a33 + 4/3a23 + 4/15$$
$$a03 + 1/15a53 + 2/5a35 + 3/5a15 - 4/5a25 - 4/25a05 - 1/25a55$$
$$a12 = 3/5a10 + 2/5a30 - 2/3a32 - 4/5a20 + 4/3a22 - 4/25a00 + 4/15a02 - 1/25a50$$
$$+ 1/15a52 + 4/15a35 + 2/5a15 - 8/15a25 - 8/75a05 - 2/75a55\,. \tag{43}$$

It is clear that this solution is *not* symmetric. To retrieve a full symmetry, we will use Relation (42) and more precisely this relation seen as

$$0 = -\frac{1}{25}q(00) + \frac{1}{5}q(20) - \frac{1}{5}q(30) + \frac{1}{25}q(50) + \frac{1}{25}q(05) - \frac{1}{25}q(55) - \frac{1}{5}q(25) + \frac{1}{5}q(35)$$
$$+ q(32) - q(22) + \frac{1}{5}q(02) - \frac{1}{5}q(52) + q(23) - q(33) - \frac{1}{5}q(03) + \frac{1}{5}q(53)\,. \tag{44}$$

where the coefficients are depicted in the following diagram:

```
 1/25   0   -1/5   1/5   0   -1/25

 0                            0

-1/5          1   [33]        1/5

 1/5         -1     1        -1/5

 0                            0

-1/25   0    1/5  -1/5   0    1/25
```

The process is made up of two steps. First, we establish the symmetry for indices $11, 14, 41, 44$ and then for the other indices. We have:

$$a11 = 4/5a10 + 22/45a30 - 14/9a32 - 38/45a20 + 22/9a22 - 74/225a00 - 38/45a02 + 4/5a01$$
$$-26/225a50 - 2/45a52 + 1/5a51 + 10/9a33 - 14/9a23 + 22/45a03 - 2/45a53 - 2/45a35$$
$$+1/5a15 - 2/45a25 - 26/225a05 + 1/225a55$$

$$+\alpha\left\{-\frac{1}{25}a00+\frac{1}{5}a20-\frac{1}{5}a30+\frac{1}{25}a50+\frac{1}{25}a05-\frac{1}{25}a55-\frac{1}{5}a25+\frac{1}{5}a35\right.$$

$$\left.+a32-a22+\frac{1}{5}a02-\frac{1}{5}a52+a23-a33-\frac{1}{5}a03+\frac{1}{5}a53\right\},$$

and

$$a41=-4/9a30+4/9a32+4/45a20+4/9a22-8/225a00-4/9a02+1/5a01+4/5a40$$

$$-92/225a50-4/9a52+4/5a51+4/9a33-8/9a23+16/45a03+4/45a53-4/9a35$$

$$+16/45a25+1/5a45-17/225a05-8/225a55$$

$$+\beta\left\{-\frac{1}{25}a00+\frac{1}{5}a20-\frac{1}{5}a30+\frac{1}{25}a50+\frac{1}{25}a05-\frac{1}{25}a55-\frac{1}{5}a25+\frac{1}{5}a35\right.$$

$$\left.+a32-a22+\frac{1}{5}a02-\frac{1}{5}a52+a23-a33-\frac{1}{5}a03+\frac{1}{5}a53\right\}.$$

Since 11 must see 00 as 41 sees 50 (and *vice versa*), we obtain the first relation

$$\alpha+\beta=2\,.$$

From this relation the other symmetries hold. Up to now, we have a family of solutions (e.g. an infinity number of solutions) depending on one parameter, α, which gives the symmetry for the four above indices. As an illustration, we obtain the following diagram for index 11 (with the scaling factor $\frac{1}{225}$ and α denoted by a):

```
-26+9a    45    -10-45a   -10+45a        0       1-9a

  0                                               0

 110-45a        -350+225a     250-225a         -10+45a

-190+45a          550-225a    -350+225a        -10-45a

 180       [11]                                  45

 -74-9a   180     -190+45a      110-45a    0      -26+9a
```

Second, we consider the symmetry for indices $21, 31, 24, 34$ and $12, 13, 42, 43$. We have

$$a21=4/15a30-4/3a32+8/3a22-8/75a00-16/15a02+3/5a01-4/25a50-4/15a52$$

$$+2/5a51+4/3a33-2a23+2/3a03-4/15a35+1/3a25-7/75a05+2/75a55$$

$$+\gamma\left\{-\frac{1}{25}a00+\frac{1}{5}a20-\frac{1}{5}a30+\frac{1}{25}a50+\frac{1}{25}a05-\frac{1}{25}a55-\frac{1}{5}a25+\frac{1}{5}a35\right.$$

$$\left.+a32-a22+\frac{1}{5}a02-\frac{1}{5}a52+a23-a33-\frac{1}{5}a03+\frac{1}{5}a53\right\},$$

$$a12=3/5a10+2/5a30-2/3a32-4/5a20+4/3a22-4/25a00+4/15a02-1/25a50$$

$$+1/15a52+4/15a35+2/5a15-8/15a25-8/75a05-2/75a55$$

$$+\delta\left\{-\frac{1}{25}a00+\frac{1}{5}a20-\frac{1}{5}a30+\frac{1}{25}a50+\frac{1}{25}a05-\frac{1}{25}a55-\frac{1}{5}a25+\frac{1}{5}a35\right.$$

$$\left.+a32-a22+\frac{1}{5}a02-\frac{1}{5}a52+a23-a33-\frac{1}{5}a03+\frac{1}{5}a53\right\}\,,$$

$$a13=2/5a10+4/15a30-8/15a20-8/75a00-2/75a50-2/3a33+4/3a23+4/15$$

$$a03+1/15a53+2/5a35+3/5a15-4/5a25-4/25a05-1/25a55$$

$$+\lambda\left\{-\frac{1}{25}a00+\frac{1}{5}a20-\frac{1}{5}a30+\frac{1}{25}a50+\frac{1}{25}a05-\frac{1}{25}a55-\frac{1}{5}a25+\frac{1}{5}a35\right.$$

$$\left.+a32-a22+\frac{1}{5}a02-\frac{1}{5}a52+a23-a33-\frac{1}{5}a03+\frac{1}{5}a53\right\}\,.$$

From this, we derive two additionnal relations to achieve the symmetry:

$$\gamma-\delta=\frac{4}{3}\quad\text{and}\quad\delta+\lambda=0\,,$$

from the last relation, the symmetry is already verified for indices 12 and 13. Hence, $\delta=\lambda=0$ and thus $\gamma=\frac{4}{3}$. Now, for these set of indices, we have a unique solution (with no parameter). As an illustration, we obtain the following diagram for index 21 (with the scaling factor $\frac{1}{75}$):

-3	0	5	0	0	-2
0					0
30		-50	0		20
-60		100	0		-40
45		[21]			30
-12	0	20	0	0	-8

Among the monomial of degree 5, the obtained Serendipity family includes u^5v and uv^5. Thus, monomials u^4v^2, u^3v^3, u^2v^4 are missing. In order to fix α, the space must be enriched with one of the previous monomials. For symmetry reasons, the only choice is to include the monomial u^3v^3 in the polynomial space.

It is easy, by means of instanciations, to see that

$$5^6\times u^3v^3=\{p^c_{11}+8p^c_{12}+27p^c_{13}+64p^c_{14}+125p^c_{15}\}+2^3\{p^c_{21}+8p^c_{22}+27p^c_{23}+64p^c_{24}+125p^c_{25}\}$$

$$+3^3\{p^c_{31}+8p^c_{32}+27p^c_{33}+64p^c_{34}+125p^c_{35}\}+4^3\{p^c_{41}+8p^c_{42}+27p^c_{43}+64p^c_{44}+125p^c_{45}\}$$

$$+5^3\{p^c_{51}+8p^c_{52}+27p^c_{53}+64p^c_{54}+125p^c_{55}\}\,,$$

then, following Relation (16), we have (as for p^c_{11})

$$5^6\times u^3v^3=\{p^c_{11}+125p^c_{15}\}+2^3\{8p^c_{22}+27p^c_{23}+125p^c_{25}\}$$

$$+3^3\{8p^c_{32}+27p^c_{33}+125p^c_{35}\}+4^3\{125p^c_{45}\}+5^3\{p^c_{51}+8p^c_{52}+27p^c_{53}+64p^c_{54}+125p^c_{55}\}\;+\;\ldots\,,$$

and in terms of p^c_{11} and omitting the factor, we only have the contributions of

$$\{p^c_{11} + 125p^c_{15}\} + 2^3\{8p^c_{22} + 27p^c_{23} + 125p^c_{25}\}$$
$$+3^3\{8p^c_{32} + 27p^c_{33} + 125p^c_{35}\} + 4^3\{125p^c_{45}\} + 5^3\{p^c_{51} + 8p^c_{52} + 27p^c_{53} + 64p^c_{54} + 125p^c_{55}\}\ ,$$

which, in terms of the coefficients, reads

$$\left\{1 - 125\alpha^{11}_{15}\right\} + 2^3\left\{-8\alpha^{11}_{22} - 27\alpha^{11}_{23} - 125\alpha^{11}_{25}\right\}$$
$$+3^3\left\{-8\alpha^{11}_{32} - 27\alpha^{11}_{33} - 125\alpha^{11}_{35}\right\} + 4^3\left\{-125\alpha^{11}_{45}\right\} + 5^3\left\{-\alpha^{11}_{51} - 8\alpha^{11}_{52} - 27\alpha^{11}_{53} - 64\alpha^{11}_{54} - 125\alpha^{11}_{55}\right\}\ ,$$

now, we replace the coefficients to obtain an equation in parameter α, which results in $\alpha = \frac{2}{3}$ and this value is also the right value to cancel all the contributions for the other (similar) indices.

To complete the proof, we check that index 21 has the same property so that, by symmetry, u^3v^3 is in the space. To this end, we compute the coefficient

$$2^3\left\{1 - 8\alpha^{21}_{22} - 27\alpha^{21}_{23} - 125\alpha^{21}_{25}\right\}$$
$$+3^3\left\{-8\alpha^{21}_{32} - 27\alpha^{21}_{33} - 125\alpha^{21}_{35}\right\} + 4^3\left\{-125\alpha^{21}_{45}\right\} + 5^3\left\{-\alpha^{21}_{51} - 8\alpha^{21}_{52} - 27\alpha^{21}_{53} - 64\alpha^{21}_{54} - 125\alpha^{21}_{55}\right\}\ ,$$

where there is no parameter but which is null as expected (so it is for the other (similar) indices).

The update of the diagram for index 11 gives (with the scaling factor $\frac{1}{45}$)

-4	9	-8	4	0	-1
0					0
16		-40	20		4
-32		80	-40		-8
36	[11]				9
-16	36	-32	16	0	-4

and we have the previous (unchanged) diagram for index $_{21}$. Since the shape functions satisfy the same relations, we find all the coefficients and, for the *type* functions, we have the following table:

	$p_{00}(u,v)$	$p_{10}(u,v)$	$p_{20}(u,v)$	$p_{22}(u,v)$
ω_{11}	-16/45	4/5	-32/45	16/9
ω_{21}	-12/75	0	20/75	100/75
ω_{31}	-8/75	0	0	0
ω_{41}	-4/45	0	16/45	-8/9
ω_{42}	-3/75	0	30/75	-50/75
ω_{43}	-2/75	0	20/75	0
ω_{44}	-1/45	0	4/45	4/9
ω_{34}	-2/75	0	0	0
ω_{24}	-3/75	0	5/75	-50/75
ω_{14}	-4/45	1/5	-8/45	-8/9
ω_{13}	-8/75	2/5	-40/75	0
ω_{12}	-12/75	3/5	-60/75	100/75

Coefficients for the repartition, ω_{kl} stands for α_{ij}^{kl}.

Using these coefficients, we have the reduced shape functions fully defined via (16), see hereafter.

Then the reduced element, seen as a patch reads

$$M(u,v) = \sum_{ij} p_{ij}(u,v) A_{ij} \,,$$

where ij denotes the indices of the (4) vertices, the (16) edge nodes and the (4) internal nodes of the reduced element, i.e., **24 indices**. We replace again the p_{ij}s by means of the p_{ij}^cs, then, with evident notations, we have

$$M(u,v) = \sum_{ij} \left\{ p_{ij}^c(u,v) + \sum_{kl} \alpha_{ij}^{kl} p_{kl}^c(u,v) \right\} A_{ij} \,,$$

with, for example, α_{ij}^{11} and α_{ij}^{21}, the coefficients in the above diagrams. Then, this reads also

$$M(u,v) = \sum_{ij} p_{ij}^c(u,v) A_{ij} + \sum_{ij} \sum_{kl} \alpha_{ij}^{kl} A_{ij} p_{kl}^c(u,v) \; = \sum_{ij} p_{ij}^c(u,v) A_{ij} + \sum_{kl} \sum_{ij} \alpha_{ij}^{kl} A_{ij} p_{kl}^c(u,v) \,,$$

therefore let

$$A_{kl} = \sum_{ij} \alpha_{ij}^{kl} A_{ij} \,,$$

again with **24 indices**, so that the patch is

$$M(u,v) = \sum_{ij} p_{ij}^c(u,v) A_{ij} \,,$$

but now with **36 indices**. In other words, we have *invented* the nodes $A_{11}, A_{21}, ...$ with which we can define a complete element fully equivalent to the reduced element. As already seen, a complete element is equivalent to the Bézier patch

$$\sum_{i=0,5} \sum_{j=0,5} B_i^5(u) B_j^5(v) P_{ij} \,,$$

from which we obtain $P_{11}, P_{21}, ...$

In practice. The 24-node quad is easy to analyse. Given its nodes, we find its edge control points by the formulae, here for edge $A_{00}A_{50}$,

$$P_{10} = \frac{-77A_{00} + 300A_{10} - 300A_{20} + 200A_{30} - 75A_{40} + 12A_{50}}{60} \,,$$

$$P_{20} = \frac{269A_{00} - 1450A_{10} + 2950A_{20} - 2300A_{30} + 925A_{40} - 154A_{50}}{240} \,,$$

$$P_{30} = \frac{-154A_{00} + 925A_{10} - 2300A_{20} + 2950A_{30} - 1450A_{40} + 269A_{50}}{240} \,,$$

$$P_{40} = \frac{12A_{00} - 75A_{10} + 200A_{20} - 300A_{30} + 300A_{40} - 77A_{50}}{60} \,.$$

Then we compute $P_{11}, P_{21}, P_{31}, P_{41}, ..., P_{12}$ by solving the corresponding system. Then, just use Relation (15) to have the control coefficients of the jacobian.

The shape functions. With the values of α_{ij}^{kl} and using Relation (16), it is easy to obtain the explicit expression of the shape functions. Indeed, only four of them must be made explicit since, for symmetry reasons, the other functions are evident to obtain. Therefore, we use Relation (10) to express $p^c_{00}(u,v)$, $p^c_{10}(u,v)$, $p^c_{20}(u,v)$, $p^c_{22}(u,v)$ and $p^c_{11}(u,v), p^c_{21}(u,v), ..., p^c_{12}(u,v)$ and then compute $p_{00}(u,v)$, $p_{10}(u,v)\, p_{20}(u,v)\, p_{22}(u,v)$, the four *type* functions, we find

1	$p_{00}(u,v) = \frac{1}{72}(v-1)(u-1)(72 - 750u - 750v + 2625u^2 + 1250uv + 2625v^2$ $-3750u^3 - 1250u^2v - 1250uv^2 - 3750v^3 + 1875u^4 + 1250u^2v^2 + 1875v^4)$
2	$p_{10}(u,v) = \frac{25}{24}u(2-5u)(3-5u)(4-5u)(1-u)(1-v)$
3	$p_{20}(u,v) = \frac{25}{36}u(5u-3)(u-1)(v-1)(12 - 75u + 25v + 75u^2 - 25v^2)$
21	$p_{22}(u,v) = \frac{625}{36}uv(5v-3)(v-1)(5u-3)(u-1)$

Type shape functions of one serendipity 24-node quad of degree 5.

Observe how simple is p_{10} and the formal beauty of the three other functions. By symmetry $(u \to 1-u, (u,v) \to (v,u), etc.)$, the full list is obtained, as examples, we have:

$$p_{30}(u,v) = p_{20}(1-u,v) \quad , \quad p_{01}(u,v) = p_{10}(v,u)\,,$$

$$p_{51}(u,v) = p_{01}(1-u,v) \quad , \quad p_{15}(u,v) = p_{10}(u,1-v)\,, etc.$$

This concludes the case of the 24-node quad.

Reduced quads of higher degree? In [1], we find serendipity quads and hexes of arbitrary order, but, as seen for the previous element, they are not fully symmetric as can be shown in the following table (for quads only) where we give the dimension of space P^d, namely $\frac{(d+1)(d+2)}{2}$, the dimension of the serendipity space, the number of internal nodes and the number of nodes to be added to recover a fully symmetric configuration.

d	$\dim(P^d)$	$\dim(P^d) + u^d v + uv^d$	$\#nodes$	$+$
2	6	8	0	0
3	10	12	0	0
4	15	17	1	0
5	21	23	3	1
6	28	30	6	2
7	36	38	10	2

Dimension of the polynomial spaces, number of internal nodes and number of "missing" nodes.

Following what we did for the 24-node quad, we think that, formally speaking, the same story applies to higher order. Nevertheless, say at order 6, the constants that can be found involve a polynomial of degree 2 (due to the D^4 derivatives) and thus, for a given sub-template, one needs to consider a combination of 3 consecutive values. To achieve a full symmetry, we think that we can invent the necessary nodes in the same way (by means of a LU factorization).

As a consequence, and, *a fortiori*, for higher order elements, the construction seems to be fastidious and we stop here our discussion.

6 Tridimensional complete Lagrange elements

Formulae (10) and (11) extend to (complete) hexahedra and (complete) simplices and give the shape functions.

6.1 Complete Lagrange tetrahedra or hexahedra of degree d

Tetrahedra of degree d. We use the definition (12) to discuss the case of a degree d tetrahedron, so we have

$$\Theta = \theta(u,v,w,t) = \sum_{i+j+k+l=d} B^d_{ijkl}(u,v,w,t) P_{ijkl}\,, \tag{45}$$

where the P_{ijkl}s are the control points. We repeat what we did for the triangle of degree d to obtain the jacobian polynomial, we find

$$\mathcal{J}(u,v,w,t) \;=\; \sum_{I+J+K+L=3(d-1)} B^q_{IJKL}(u,v,w,t) N_{IJKL}\,,$$

where $q = 3(d-1)$ and the coefficients N_{IJKL} are

$$N_{IJKL} \;=\; d^3 \sum_{|i|=I,|j|=J,|k|=K,|l|=L} \frac{C^{d-1}_{i_1j_1k_1l_1} C^{d-1}_{i_2j_2k_2l_2} C^{d-1}_{i_3j_3k_3l_3}}{C^q_{IJKL}} \,|\, \Delta^{1000}_{i_1+1,j_1k_1l_1} \quad \Delta^{0100}_{i_2+1,j_2k_2l_2} \quad \Delta^{0010}_{i_3+1,j_3k_3l_3} \,|\,, \tag{46}$$

where $|i| = i_1 + i_2 + i_3, \ldots$ and with $i_1 + j_1 + k_1 + l_1 = i_2 + j_2 + k_2 + l_2 = i_3 + j_3 + k_3 + l_3 = d-1$ and with the following Δ:

$$\Delta^{1000}_{ijkl} = \overrightarrow{P_{ijkl}P_{i-1,j+1,k,l}}, \quad \Delta^{0010}_{ijkl} = \overrightarrow{P_{ijkl}P_{i-1,j,k+1,l}} \text{ and } \quad \Delta^{0010}_{ijkl} = \overrightarrow{P_{ijkl}P_{i-1,j,k,l+1}}\,.$$

The jacobian is a polynomial of degree $q = 3(d-1)$, the number of control coefficients is $\frac{(q+1)(q+2)}{2}$ and the number of determinants involved in these coefficients is $(\frac{d\times(d+1)}{2} + \frac{(d-1)d}{2} + \ldots)^3$.

Hexahedra of degree d. We use the definition (14) to discuss the case of a degree d hexahedron, so we have

$$\theta(u,v,w) \;=\; \sum_{i=0,d} \sum_{j=0,d} \sum_{k=0,d} B^d_i(u) B^d_j(v) B^d_k(w)\, P_{ijk}\,, \tag{47}$$

and, the jacobian polynomial has the form

$$\mathcal{J}(u,v,w) \;=\; \sum_{I=0,q} \sum_{J=0,q} \sum_{K=0,q} B^q_I(u) B^q_J(v) B^q_K(w) N_{IJK}\,,$$

where $q = 3d-1$ and the coefficient N_{IJK}s are

$$N_{IJK} \;=\; d^3 \sum_{|i|=I} \sum_{|j|=J} \sum_{|k|=K} \frac{C^{d-1}_{i_1} C^d_{i_2} C^d_{i_3}}{C^q_I} \frac{C^d_{j_1} C^{d-1}_{j_2} C^d_{j_3}}{C^q_J} \frac{C^d_{k_1} C^d_{k_2} C^{d-1}_{k_3}}{C^q_K} |\Delta^{100}_{i_1,j_1,k_1} \quad \Delta^{010}_{i_2,j_2,k_2} \quad \Delta^{001}_{i_3,j_3,k_3}|$$

$$\text{with } \Delta^{100}_{ijk} = \overrightarrow{P_{ijk}P_{i+1,jk}}\,, \Delta^{010}_{ijk} = \overrightarrow{P_{ijk}P_{i,j+1,k}} \text{ and } \Delta^{001}_{ijk} = \overrightarrow{P_{ijk}P_{ij,k+1}}\,, \tag{48}$$

where $|i| = i_1 + i_2 + i_3, \ldots$ and with $i_1 = 0, d-1, i_2 = 0, d, i_3 = 0, d, j_1 = 0, d, j_2 = 0, d-1, j_3 = 0, d, k_1 = 0, d, k_2 = 0, d, k_3 = 0, d-1$. The jacobian is a polynomial of degree $q \times q \times q = (3d-1) \times (3d-1) \times (3d-1)$ and the number of control coefficients is $(q+1)^3$.

7 Tridimensional incomplete Lagrange elements

First of all, formulae (16) and (17) extend to hexahedra and simplices and show how the reduced shape functions are related to the complete shape functions. Then, some of the complete Lagrange elements have their related incomplete elements. The method to constructing those reduced elements is basically the same, e.g. by means of Taylor expansions while taking into account what polynomial space we like to have.

7.1 Tetrahedra

The order 3 tetrahedron. The complete element has 20 nodes with one node per face and no node inside the volume. We propose to apply to the faces what we did for the order 3 triangle. In other words, we apply Relation (18), to the faces. For the triangle, we had

$$12q(\hat{A}_{111}) + 2\sum_{ijk\in\mathcal{V}} q(\hat{A}_{ijk}) - 3\sum_{ijk\in\mathcal{E}} q(\hat{A}_{ijk}) = 0\,,$$

and this leads to four relations, for instance for face $t = 0$, we simply have (with evident notations)

$$12q(\hat{A}_{1110}) + 2\sum_{ijkl\in\mathcal{V}_{t=0}} q(\hat{A}_{ijk0}) - 3\sum_{ijkl\in\mathcal{E}_{t=0}} q(\hat{A}_{ijk0}) = 0\,,$$

and similar relations for the other faces where the template is centered at $\hat{A}_{1101}, \hat{A}_{1011}, \hat{A}_{0111}$ for the faces $w = 0, v = 0$ and $u = 0$, respectively. Therefore we have a system of 4 equations.

Using the extension of Relation (17), we look for reduced functions, polynomials like

$$p_{ijkl}(u,v,w,t) = p^c_{ijkl}(u,v,w,t) + \alpha p^c_{1110}(u,v,w,t) + \beta p^c_{1101}(u,v,w,t) + \gamma p^c_{1011}(u,v,w,t) + \delta p^c_{0111}(u,v,w,t)\,.$$

Due to the symmetry, there is only two *type* shape functions, $p_{3000}(u,v,w,t)$ and $p_{2100}(u,v,w,t)$. In the four above equations, we replace the generic function q by p_{3000}, then we immediately have $\alpha = \beta = \gamma = -\frac{1}{6}$ and $\delta = 0$, the face opposite to $\hat{A}_{3000}$ has no contribution to p_{3000}.

To define p_{2100}, we use the same method and we find $\alpha = \beta = \frac{1}{4}$ and $\gamma = \delta = 0$, the two faces opposite to edge $\hat{A}_{3000}\hat{A}_{0300}$ have no contribution to p_{2100}.

The shape functions. With the values of α, β, γ and δ and using Relation (17), extended to three dimensions, it is easy to obtain the explicit expression of the shape functions. Indeed, only two of them must be explicited since, for symmetry reasons, the other functions are evident to obtain. Therefore, we use a relation like Relation (11) to express $p^c_{3000}(u,v,w,t), p^c_{2100}(u,v,w,t)$ and $p^c_{1110}(u,v,w,t), p^c_{1110}(u,v,w,t), p^c_{1110}(u,v,w,t), p^c_{1110}(u,v,w,t)$, and then compute $p_{3000}(u,v,w,t)$ and $p_{2100}(u,v,w,t)$, the two *type* functions, we find

$$p_{3000}(u,v,w,t) = \frac{1}{2}u(2u^2 - 5uv - 5uw - 5ut + 2v^2 - 5vw - 5vt + 2w^2 - 5wt + 2t^2)\,,$$

$$\text{or}\quad p_{3000}(x,y,z) = \frac{1}{2}(1-x-y-z)(2 - 9x - 9y - 9z + 9x^2 + 9xy + 9xz + 9y^2 + 9yz + 9z^2)\,,$$

$$\text{and}\quad p_{2100}(u,v,w,t) = \frac{9}{4}uv(4u - 2v + w + t)\,,$$

$$\text{or}\quad p_{2100}(x,y,z) = \frac{9}{4}(1-x-y-z)x(4 - 6x - 3y - 3z)\,.$$

By symmetry $(x \to 1-x, (u,v,w,t) \to (v,u,w,t), etc., in \quad x \quad or \quad u)$, we have the full list.

1	$p_{3000}(x,y,z)=\frac{9}{2}(1-x-y-z)(\frac{2}{9}-x-y-z+xy+xz+yz+x^2+y^2+z^2)$
4	$p_{2100}(x,y,z)=\frac{9}{4}(1-x-y-z)x(4-6x-3y-3z)$

Type shape functions of the 16-node tetrahedron of degree 3.

In practice. The 16-node tetrahedron is easy to analyse. Given its nodes, we find its edge control points by the formulae (see the case of the triangle of degree 3) and then we compute $P_{1110}, P_{1101}, P_{1011}, P_{0111}$ using, by face, the formula of the triangle. Then, with these points, the element reads

$$\theta(u,v,w,t)=\sum_{i+j+k+l=3} B^3_{ijkl}(u,v,w,t)P_{ijkl}\,,$$

and it is just required to use Relation (46) to have the control coefficients of the jacobian.

This concludes the case of the 16-node tetrahedron.

The order 4 tetrahedron. The complete element has 35 nodes with three nodes per face and one node inside the volume. We propose to apply to the faces what we did for the order 4 triangle. In other words, we will apply Relations (22) to the faces of the element to deal with their three nodes and to account for the volume node, we will apply Relation (26) to four adequately defined pseudo-faces. Therefore, using Relations (22). e.g.,

$$15q(\hat{A}_{211})-15q(\hat{A}_{121})=-2q(\hat{A}_{400})+2q(\hat{A}_{040})$$

$$-5q(\hat{A}_{130})-2q(\hat{A}_{103})+5q(\hat{A}_{310})+3q(\hat{A}_{202})+3q(\hat{A}_{301})+2q(\hat{A}_{013})-3q(\hat{A}_{031})-3q(\hat{A}_{022})\,,$$

and

$$15q(\hat{A}_{211})-15q(\hat{A}_{112})=-2q(\hat{A}_{400})+2q(\hat{A}_{004})$$

$$-2q(\hat{A}_{130})-5q(\hat{A}_{103})+3q(\hat{A}_{310})+3q(\hat{A}_{220})+5q(\hat{A}_{301})+2q(\hat{A}_{031})-3q(\hat{A}_{022})-3q(\hat{A}_{013})\,,$$

we mechanically obtain, for the face $t=0$, the two following equations

$$(R1)\qquad 15q(\hat{A}_{2110})-15q(\hat{A}_{1210})=-2q(\hat{A}_{4000})+2q(\hat{A}_{0400})$$

$$-5q(\hat{A}_{1300})-2q(\hat{A}_{1030})+5q(\hat{A}_{3100})+3q(\hat{A}_{2020})+3q(\hat{A}_{3010})+2q(\hat{A}_{0130})-3q(\hat{A}_{0310})-3q(\hat{A}_{0220})\,,$$

and

$$(R2)\qquad 15q(\hat{A}_{2110})-15q(\hat{A}_{1120})=-2q(\hat{A}_{4000})+2q(\hat{A}_{0040})$$

$$-2q(\hat{A}_{1300})-5q(\hat{A}_{1030})+3q(\hat{A}_{3100})+3q(\hat{A}_{2200})+5q(\hat{A}_{3010})+2q(\hat{A}_{0310})-3q(\hat{A}_{0220})-3q(\hat{A}_{0130})\,,$$

which correspond to the following template:

```
                0040

             1030    0130

          2020    1120    0220

       3010    2110    1210    0310

    4000    3100    2200    1300    0400          (t=0)
```

The same applies (after a permutation in the indices to account for the templates in hand) to faces $w = 0, v = 0$ and $u = 0$, resulting in 8 equations.

To complete this system, we have to find the equations giving the relations between the volume node, $\hat{A}_{1111}$, and the nodes in the boundary of the element. To this end, we consider the pseudo-faces defined by the planes $t = \frac{1}{4}, w = \frac{1}{4}, v = \frac{1}{4}$ and $u = \frac{1}{4}$ and we use Relation (26) in the corresponding templates. As an example, the template for $t = \frac{1}{4}$ is depicted in the following diagram:

```
          0031

       1021    0121
                                        (t=1/4)
    2011    1111    0211

 3001    2101    1201    0301
```

and the relation

$$C_{211} \;=\; 12\, q(\hat{A}_{211}) \,+\, 2 \sum_{ijk \in \mathcal{V}_{211}} q(\hat{A}_{ijk}) \,-\, 3 \sum_{ijk \in \mathcal{E}_{211}} q(\hat{A}_{ijk})\,,$$

reads for the above template (with evident notations)

$$C_{1111-t} = 12q(A_{1111}) + 2q(A_{3001}) + 2q(A_{0301}) + 2q(A_{0031})$$

$$-3q(A_{2101}) - 3q(A_{1201}) - 3q(A_{0211}) - 3q(A_{0121}) - 3q(A_{1021}) - 3q(A_{2011})\,.$$

Similarly, we obtain 3 more equations related to the three other pseudo-faces. To summarize, we have 12 equations, meaning that we have to consider the volume node as a node of the reduced element, in other words, we have only a chance of canceling the 12 face nodes.

In [19], we give the explicit reading of the 12×12 matrix of the system and we see that the determinant of this matrix is null. To better understand this point, using the extension of Relation (17), we look for reduced functions, polynomials like

$$p_{ijkl}(u,v,w,t) = p^c_{ijkl}(u,v,w,t) + \sum_{mnop} \alpha^{mnop}_{ijkl} p^c_{mnop}(u,v,w,t)\,,$$

we fixe index $_{ijkl} =_{4000}$ and we write the 12 corresponding equations. Omitting index ^{mnop}, we find the following relations:

$$(R1) \qquad 15\alpha_{2110} - 15\alpha_{1210} = -2$$

$$(R2) \qquad 15\alpha_{2110} - 15\alpha_{1120} = -2$$

$$(R3) \qquad 15\alpha_{2101} - 15\alpha_{1201} = -2$$

$$(R4) \qquad 15\alpha_{2101} - 15\alpha_{1102} = -2$$

$$(R5) \qquad 15\alpha_{2011} - 15\alpha_{1021} = -2$$

$$(R6) \qquad 15\alpha_{2011} - 15\alpha_{1012} = -2$$

$$(R7) \qquad 15\alpha_{0211} - 15\alpha_{0121} = 0$$

$$(R8) \qquad 15\alpha_{0211} - 15\alpha_{0112} = 0$$

$$(R9) \qquad -3\alpha_{2101} - 3\alpha_{1201} - 3\alpha_{0211} - 3\alpha_{0121} - 3\alpha_{1021} - 3\alpha_{2011} + 3\alpha_{1210} + 3\alpha_{1120} - 12\alpha_{2110} = 2$$

$$(R10) \qquad -3\alpha_{2110} - 3\alpha_{1210} - 3\alpha_{0211} - 3\alpha_{0112} - 3\alpha_{1012} - 3\alpha_{2011} + 3\alpha_{1201} + 3\alpha_{1102} - 12\alpha_{2101} = 2$$

$$(R11) \qquad -3\alpha_{2110} - 3\alpha_{1120} - 3\alpha_{0121} - 3\alpha_{0112} - 3\alpha_{1102} - 3\alpha_{2101} + 3\alpha_{1021} + 3\alpha_{1012} - 12\alpha_{2011} = 2$$

$$(R12) \qquad -3\alpha_{1210} - 3\alpha_{1120} - 3\alpha_{1021} - 3\alpha_{1012} - 3\alpha_{1102} - 3\alpha_{1201} + 3\alpha_{0121} + 3\alpha_{0112} - 12\alpha_{0211} = 0\,.$$

These relations imply that

$$(R9) \qquad \alpha_{1102} + \alpha_{0112} + \alpha_{1012} + \alpha_{1120} = \frac{1}{15}\,,$$

together with

$$(R12) \qquad \alpha_{1120} + \alpha_{1012} + \alpha_{1102} + \alpha_{0112} = 0\,,$$

which is *impossible*. As a conclusion, we don't find a solution, using our method (Taylor), there is no way to constructing a reduced order 4 tetrahedron covering space P^3. This negative result is not a surprise since we had the same conclusion for the order 4 triangle.

As we did for the triangle, we limit our quest so as to only cover the space P^2. The conclusion is similar, we don't find a solution since we have one inconsistency with the "central" node. The order 4 tetrahedron is too rigid.

Therefore, we stop here our study about tetrahedra while the question to know if this negative issue holds for higher order tetrahedra remains.

7.2 Hexahedra

The idea is the same, we extend to the faces and, if needed, to some pseudo-faces, the relations obtained for the quadrilaterals.

The order 2 hexahedron. The complete element is a 27-node hex with one node per face and one node in the volume. We consider Relation (30), e.g.,

$$4q(\hat{A}_{11}) + \sum_{ij\in\mathcal{V}} q(\hat{A}_{ij}) - 2\sum_{ij\in\mathcal{E}} q(\hat{A}_{ij}) = 0\,,$$

and we write this relation for the six faces of the element. The face $w = 0$ is depicted in the following diagram

```
020   120   220

010   110   210

000   100   200
```

and we have a first equation

$$4q(\hat{A}_{110}) + \Big\{q(\hat{A}_{000}) + q(\hat{A}_{200}) + q(\hat{A}_{220}) + q(\hat{A}_{020})\Big\} - 2\left\{q(\hat{A}_{100}) + q(\hat{A}_{210}) + q(\hat{A}_{120}) + q(\hat{A}_{010})\right\} = 0\,.$$

By applying the same relation to the five other faces (and the appropriate templates in terms of indices), we obtain five more equations. Now we provide a relation between the central node ($\hat{A}_{111}$) and the nodes on the boundary of the element by considering the following template corresponding to the plane $u = \frac{1}{2}$ and the pseudo-face so defined

```
102    112    122

101    111    121

100    110    120
```

and then we add a relation involving the central node

$$4q(\hat{A}_{111})+\Big\{q(\hat{A}_{100})+q(\hat{A}_{102})+q(\hat{A}_{120})+q(\hat{A}_{122})\Big\}-2\Big\{q(\hat{A}_{101})+q(\hat{A}_{121})+q(\hat{A}_{110})+q(\hat{A}_{112})\Big\}=0\,, \tag{49}$$

and those 7 relations define the serendipity property of the element.

Then we write the values at the face nodes in this last relation by their expressions in the their own face, it results equation

$$4q(\hat{A}_{111})+\sum_{ijk\in\mathcal{V}} q(\hat{A}_{ijk})-\sum_{ijk\in\mathcal{E}} q(\hat{A}_{ijk})=0\,, \tag{50}$$

which links the central value with the value at the vertices and the edge nodes.

The reduced shape functions follow the form of Relation (16), namely we have

$$p_{ijk}(u,v,w)=p^c_{ijk}(u,v,w)+\sum_{lmn}\alpha^{lmn}_{ijk}p^c_{lmn}(u,v,w)\,,$$

and these functions satisfy all the above equations. For symmetry reasons, there is only two *type* functions so we fix $ijk = 000$ and $ijk = 100$ to obtain the coefficients of repartition from the initial system. The solution is

$$\alpha^{110}_{000}=\alpha^{101}_{000}=\alpha^{011}_{000}=\alpha^{111}_{000}=-\frac{1}{4}\,,$$

$$\text{and}\quad \alpha^{110}_{100}=\alpha^{101}_{100}=\frac{1}{2}\quad\text{and}\quad \alpha^{111}_{100}=\frac{1}{4}\,,$$

one can observe that the "corner" shape function only depends on the complete shape functions of its three incident faces and the central one. For an edge shape function, only the central function and the functions of the two incident faces contribute.

With the values of the coefficients and following the generic form of the functions, we have

1	$q_{000}(u,v,w)=(1-u)(1-v)(1-w)(1-2u-2v-2w)$
9	$q_{100}(u,v,w)=4u(1-u)(1-v)(1-w)$

Type shape functions of the 20-node hexahedron of degree 2.

In practice. The 20-node hexahedron is easy to analyse. Given its nodes, we find its edge control points by the formulae (see the case of the quad of degree 2) and then we compute the control points of the face using, by face, the formula of the quadrilateral. Then, with these points, the element reads

$$\theta(u,v,w)=\sum_{i=0,2}\sum_{j=0,2}\sum_{k=0,2}B^2_i(u)B^2_j(v)B^2_k(w)P_{ijk}\,,$$

from which we compute the last control point, P_{111}, we have

$$P_{111} = -\frac{5}{8}\sum_{ijk\in\mathcal{V}} P_{ijk} + \frac{3}{4}\sum_{ijk\in\mathcal{E}} P_{ijk} - \frac{1}{2}\sum_{ijk\in\mathcal{F}} P_{ijk}\,,$$

i.e., P_{111} is written in terms of the vertices, the control points of the edges (other than the vertices) and the control points of the faces (other than all the previous). To end, it is just required to use Relation (48) to have the control coefficients of the jacobian.

About the polynomial space of the reduced element. The diagram of the polynomial space is

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & u & v & w & & \\
u^2 & uv & uw & v^2 & vw & w^2 & \\
u^2v & u^2w & uvw & uv^2 & v^2w & uw^2 & vw^2 \\
u^2vw & & & uv^2w & & & uvw^2
\end{array}$$

where we have space P^2, the monomials of degree $d+1=3$ with at least one variable linearly involved and the the monomials of degree $d+2=4$ with two variables linearly involved, therefore 20 monomials.

This concludes the case of the 20-node hexahedron (note that this element can be defined by means of a transfinite interpolation, [23]).

The order 3 hexahedron. The complete element is a 64-node hex with four nodes per face and eight nodes in the volume. The number of nodes we want to cancel is 32 ($6\times 4+8=32$). We apply to the nodes of the faces what we did for the quadrilateral of degree 3 where we have 4 equations, therefore this lead to 24 equations. To have 8 equations more we consider two planes (for instance $u=\frac{1}{3}$ and $u=\frac{2}{3}$) and consider the corresponding templates. As a result we have 32 equations for 32 unknowns. We proceed these equations as we did in the previous element so as to find the links between the volume values (of a generic function) and the values on the boundary (vertices, edge nodes and face nodes) of the elements.

The reduced shape functions follow the form of Relation (16), namely we have

$$p_{ijk}(u,v,w) = p^c_{ijk}(u,v,w) + \sum_{lmn} \alpha^{lmn}_{ijk} p^c_{lmn}(u,v,w)\,,$$

and these functions satisfy all the above equations. For symmetry reasons, there is also only two *type* functions so we fix $ijk=000$ and $ijk=100$ to obtain the coefficients of repartition from the initial system. The solution, [20], is given by means of a table and some diagrams. The coefficients for q_{000} are listed in the following table (where we omit the index $_{000}$):

$\alpha^{110} = -\frac{4}{9}$	$\alpha^{210} = -\frac{2}{9}$	$\alpha^{120} = -\frac{2}{9}$	$\alpha^{220} = -\frac{1}{9}$
$\alpha^{011} = -\frac{4}{9}$	$\alpha^{021} = -\frac{2}{9}$	$\alpha^{012} = -\frac{2}{9}$	$\alpha^{022} = -\frac{1}{9}$
$\alpha^{101} = -\frac{4}{9}$	$\alpha^{201} = -\frac{2}{9}$	$\alpha^{102} = -\frac{2}{9}$	$\alpha^{202} = -\frac{1}{9}$
$\alpha^{122} = -\frac{4}{27}$	$\alpha^{222} = -\frac{2}{27}$	$\alpha^{121} = -\frac{8}{27}$	$\alpha^{221} = -\frac{4}{27}$
$\alpha^{112} = -\frac{8}{27}$	$\alpha^{212} = -\frac{4}{27}$	$\alpha^{111} = -\frac{16}{27}$	$\alpha^{211} = -\frac{8}{27}$

As for the nodes in the plane $w=0$, the coefficients are

```
   .        .        .    .

   .       120      220   .              -2/9    -1/9

   .       110      210   .              -4/9    -2/9

[000]        .        .    .      [000]
```

in the plane $w = \frac{1}{3}$, we have

```
   .        .        .    .

  021      121      221   .        -2/9     -8/27   -4/27

  011      111      211   .        -4/9    -16/27   -8/27

[000]      101      201    .      [000]     -4/9    -2/9
```

and in the plane $w = \frac{2}{3}$, we have

```
   .        .        .    .

  022      122      222   .        -1/9     -4/27   -2/27

  012      112      212   .        -2/9     -8/27   -4/27

[000]      102      202    .      [000]     -2/9    -1/9
```

one can observe the perfect symmetry and the mechanism of construction of these coefficients and see that all the nodes contributing to q_{000} are located on the planes $w = 0$, $w = \frac{1}{3}$ and $w = \frac{2}{3}$.

The coefficients for q_{100} are depicted in the following diagram:

```
   .        .        .    .

  102      112      122   .        1/3     2/9    1/9

  101      111      121   .        2/3     4/9    2/9

[100]      110      120   .       [100]    2/3    1/3
```

again, one can observe the perfect symmetry and the mechanism of construction of these coefficients and see that all the nodes contributing to q_{100} are located on the plane $u = \frac{1}{3}$.

With the values of the coefficients and following the generic form of the functions, we have

1	$q_{000}(u, v, w) = \frac{9}{2}(1-u)(1-v)(1-w)(\,\frac{2}{9} - v - w - u + u^2 + v^2 + w^2\,)$
9	$q_{100}(u, v, w) = \frac{9}{2}u(1-u)(2-3u)(1-v)(1-w)$

Type shape functions of the 32-node hexahedron of degree 3.

In practice. The 32-node hexahedron is easy to analyse. Given its nodes, we find its edge control points by the formulae (see the case of the quad of degree 3) and then we compute the control points of the face using, by face, the formula of the quadrilateral. Then, with these points, the element reads

$$\theta(u,v,w) = \sum_{i=0,3} \sum_{j=0,3} \sum_{k=0,3} B_i^3(u) B_j^3(v) B_k^3(w) P_{ijk}\,,$$

from which we compute the internal control points. To this end, we first define the internal nodes and then we solve the system

$$A_{111} = \theta(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

$$...$$

$$A_{222} = \theta(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})\,,$$

from which we compute $P_{111}, ..., P_{222}$. To end, it is just required to use Relation (48) to have the control coefficients of the jacobian.

About the polynomial space of the reduced element. The diagram of the polynomial space is

$$\begin{array}{c}
1 \\
u \quad v \quad w \\
u^2 \quad uv \quad uw \quad v^2 \quad vw \quad w^2 \\
u^3 \quad u^2v \quad u^2w \quad uv^2 \quad uvw \quad uw^2 \quad v^3 \quad v^2w \quad vw^2 \quad vw^3 \\
u^3v \quad u^3w \quad u^2vw \quad uv^2w \quad uvw^2 \quad uv^3 \quad uw^3 \quad v^3w \quad uw^3 \\
u^3vw \qquad\qquad uv^3w \qquad\qquad uvw^3
\end{array}$$

where we have space P^3, the monomials of degree $d+1=4$ with at least one variable linearly involved and the the monomials of degree $d+2=5$ with two variables linearly involved, therefore 32 monomials.

This concludes the case of the 32-node hexahedron.

Higher order hexahedra? The complete element of degree 4 is a 125-node hex with 9 nodes per face and 27 nodes in the volume. After [1], the serendipity hex has 50 nodes, therefore the number of nodes in the volume is $50 - 8 - 12 \times 3 - 6 \times 1 = 0$, so that we have no problem with the symmetry in the volume. Now, the number of nodes to cancel is $6 \times 9 - 6 + 27 = 75$ because one node per face is retained (as for the quad of degree 4). Therefore we need to find 75 equations. The equations are obtained by using Taylor expansions in a number of planes and in specific for those related to the 6 faces (and we return to the case of the bidimensional quad).

This seems to be rather technical and tedious so we don't like to continue the story for higher degrees, we just like to give some remarks about degrees greater or equal to 5.

The complete element of degree 5 is a 216-node hex with 16 nodes per face and 64 nodes in the volume. Again after [1], the serendipity hex has 74 nodes, therefore the number of nodes in the volume is $74 - 8 - 12 \times 4 - 6 \times 3 = 0$, so that we have no problem with the symmetry in the volume. Now, the number of nodes to cancel is $6 \times 16 - 6 \times 3 + 64 = 142$ meaning that 3 nodes per face are retained and we need to find 142 equations, or, if not, to retains a number of nodes inside the volume. Nevertheless, as pointed out before in this paper and also in [15], if one keeps the nodes distinct, it is not possible to arrange them in a completely symmetric way when $d \geq 5$. This is why we proposed the quad of degree 5 previously discussed where we have 4 nodes per face, so that *i*) a symmetric arrangement is possible and *ii*) there is no problem of conformity from one element to its six neighbors (by face) when considering the case of hexes. After this trick, we thing that it is possible to consider such elements as the others by using Taylor expansions.

In conclusion, we were not capable (or we have not enough courage) to explicitly construct reduced hexes with a degree greater than 3 and this concludes our discussion about the hexes and gives room for further works.

7.3 Other elements

Among the elements in three dimensions, we also have the prisms (pentahedra) and the pyramids. The complete prism of degree 2 has 18 nodes including one node per quadrilateral face. The face are quad of degree 2 so that the "central" node can be cancelled and we obtain a 15-node prism, see [17]. For the degree 3, the 40-node prism, one can expect that the 12 nodes on the quadrilateral faces (3×4) and the 2 nodes on the triangular faces (2×1) together with the 2 nodes in the volume can be cancelled.

Pyramidal elements are, somehow, controversial elements, for which we have various (and possibly antagonist) definitions. In [18], we proposed a definition of a pyramid as being a degenerated form of an hexahedron. With such a definition, one finds the counterpart of the serendipity hexa of degree 2, a 13-node pyramid.

8 Conclusion

In this paper, we gave a detailed discussion about the construction of some reduced elements and, this being done and given a reduced element in a mesh, we discuss the conditions that give guarantees about its geometric validity. As seen, the main idea is, given such an element in a mesh, to return to a complete element equivalent to this reduced one and then to apply what we did previously for complete elements. It turns out that this leads to properly *invent* the "missing" nodes and the "missing" control points.

Incomplete or reduced elements have a reduced number of nodes (typically the edge nodes are those of the complete elements while the number of internal nodes is zero or smaller than that in the complete element). The polynomial space is also of a smaller dimension as compared with the complete space. These two facts make attractive the reduced elements since the corresponding computational cost is, in turn, widely reduced. The following table gives some examples of the reduction obtained by reporting the gain in terms of the number of nodes.

d	2	3	3	4	4	5	2	3	3	4
geometry	quad	tria	quad	tria	quad	quad	hexa	tetra	hexa	hexa
#nodes	9	10	16	15	25	36	27	20	64	125
red. #nodes	8	9	12	12	17	24	20	17	32	50
gain	1	1	4	3	8	12	7	3	32	75
%	11%	10%	25%	20%	32%	33%	26%	15%	50%	60%

Reduction in the number of nodes for the reduced elements.

As a matter of fact, our study reveals a number of points or open questions. Indeed, we don't find any triangle of degree greater than 4, we meet a difficulty with the quads of degree greater than 4 where the complete symmetry in the nodes arrangement was an issue. In three dimension, we don't find any tetrahedron of degree greater than 3 and we were not capable to explicitly construct reduced hexes with a degree greater than 3, such a task being, at least, too technical.

The elements we find are serendipity elements as far as we consider the quads and the hexes. For the few triangles and the tetrahedron discussed in this paper, the term reduced (or incomplete) seems to be more appropriate to qualify such elements.

As regards the geometric validity (positive jacobians) of the reduced elements, we showed that the computational effort could be costly since a large number of determinants must be evaluated apart if one considers straight-sided elements.

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